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ON MODELS

OF

CUBIC SURFACES

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OF

CUBIC SURFACES

BY

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PREFACE.

ABOUT ten years ago my attention was drawn by a pamphlet of Mr H. M. Taylor to the possibility of arranging the twenty-seven straight lines on a cubic surface in position by geometrical construction, and hence to determine the form of the surface.

After constructing several models, I did not continue the series, for I subsequently found that a complete set had been made in Germany, and had been exhibited by Klein at the Chicago Exhibition. Copies of these models can be purchased.

Still the models described in this book are sufficient to give an idea of the shape of a cubic surface, and to illustrate the changes that take place under certain conditions.

The object of this book is to give an outline of analytical and geometrical methods that are used in treating of cubic surfaces, not taking the more advanced part of the subject, but considering mainly anything that may help to the construction of models.

The latter part of the book is devoted to a description of the shapes of the surfaces.

The Cambridge Philosophical Society and *The Quarterly Mathematical Journal* have kindly published many papers of mine upon the subject, including some geometrical propositions to be found in the third chapter. Many of the diagrams have already appeared in the above-mentioned journals.

The text-books on Solid Geometry by Frost and Salmon furnish most of the facts that are necessary to form equations of cubic surfaces, but in the works of Cayley the subject is most completely considered from an analytical point of view.

Reye has given a series of lectures on Geometry of Position in which cubic surfaces are treated by methods of projection. The first part of these lectures has appeared in an English translation by Thos. Holgate, and a very good summary of his methods is given in the *Encyclopædia Britannica* under the heading Geometry. The particular lectures on Cubic Surfaces are to be found in a French translation published many years ago.

In addition to the books already referred to I have quoted largely from Klein, Zeuthen, and indirectly from Sturm, chiefly from English or French translations of these authors.

The equations, a list of which is given in the first chapter, are taken from a paper of Dr Schläfli published by Dr Cayley.

W. H. B.

July 1905.

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INTRODUCTION.

A CUBIC surface may be represented by an equation of the third degree in Cartesian coordinates. If this equation be expressed in the form $uvw + xyz = 0$ where x, y, z, u, v, w are expressions involving the coordinates in the first degree, then $u = 0$, $v = 0$, $w = 0$ ...... are triple tangent planes to the surface, and any straight lines represented by the equations $u = 0$, $x = 0$; $u = 0$, $y = 0$; ...... are straight lines on the surface.

A plane such as $u = 0$ is a triple tangent plane because it touches the surface at the three angular points of the triangle in which the three planes $x = 0$, $y = 0$, $z = 0$ meet it. The general theory of surfaces is fully discussed in Dr Salmon's *Geometry of Three Dimensions* (Chapter XI.), and Dr Frost also supplies all necessary information on this subject in his *Text-book on Solid Geometry*.

When Dr Schläfli investigated the number of triple tangent planes and straight lines on a cubic surface, the question resolved itself into one of algebraical transformation, namely, In how many ways can any equation of the third degree be put into the form $uvw + xyz = 0$? Further, in how many of these transformations are u, v, w, x, y, z real, and in how many are they imaginary?

It is clear that any equation of the third degree can be expressed in the form $uvw + xyz = 0$, for by equating coefficients we obtain the necessary equations to find the coefficients of the variables in the functions u, v, w ...... and no more.

To determine the reality of the triple tangent planes Dr Schläfli used the following method. [*Phil. Trans.* Vol. 153.]

If we assume $\mathsf{A}u + \mathsf{B}v + \mathsf{C}w + \mathsf{D}x + \mathsf{E}y + \mathsf{F}z = 0$, and in this expression equate the coefficients of the variable coordinates, and the constant term to zero, we find only four equations between the six unknown quantities A, B, C, D, E, F. As the ratios of these quantities only are required, if we suppose the four equations used to express C, D, E, F in terms of A and B, we only require one condition more to express the ratio of A to B.

We remember that this condition is to be selected so that the equation $uvw + xyz = 0$ may be expressed in another of the same form.

The condition is suggested by the identical equation

$$\mathsf{A}u(\mathsf{B}v + \mathsf{D}x)(\mathsf{C}w + \mathsf{D}x) + \mathsf{D}x(\mathsf{A}u + \mathsf{E}y)(\mathsf{A}u + \mathsf{F}z) = \mathsf{ABC}uvw + \mathsf{DEF}xyz$$

and is that $\mathsf{ABC} = \mathsf{DEF}$.

This condition gives us a cubic equation, one root of which is real, while the other two may be real or imaginary.

If this condition be granted we may take

$$\mathsf{A}u(\mathsf{B}v + \mathsf{D}x)(\mathsf{C}w + \mathsf{D}x) + \mathsf{D}x(\mathsf{A}u + \mathsf{E}y)(\mathsf{A}u + \mathsf{F}z) = 0 \text{ ......} (\alpha)$$

as the equation to the surface.

We then find $\mathsf{B}v + \mathsf{D}x = 0$, $\mathsf{C}w + \mathsf{D}x = 0$, $\mathsf{A}u + \mathsf{E}y = 0$ and $\mathsf{A}u + \mathsf{F}z = 0$ as triple tangent planes to the surface.

If we take other similar arrangements for equation (α) and substitute in turn the different roots of the cubic equation $\mathsf{ABC} = \mathsf{DEF}$ we obtain twenty-seven different forms of the equation to the surface. The tangent planes are not all different in each case, there are forty-five in all.

The only necessary conditions that these planes should be real are that u, v, w, x, y, z should be real, and that $\mathsf{ABC} = \mathsf{DEF}$ should have real roots.

If any one of the quantities is imaginary, since the equations are real, we must have another quantity imaginary and conjugate to it: therefore if any triple tangent plane, or straight line on the surface is imaginary, there must be another which is imaginary and may be regarded as conjugate to it.

A list of the different cases is given on page 2.

The cubic surface is of the third order, for a straight line meets it in three points; it is also of the twelfth class. The class of a surface is fixed by the number of tangent planes that can be drawn to it, which all pass through a fixed arbitrary straight line.

The existence of nodes on the surface and their character is examined by Dr Salmon in his *Geometry of Three Dimensions* very simply as follows (Chapter XI.).

Write the equation to the surface as $u_0 + u_1 + u_2 + u_3 = 0$ where u_0, u_1, u_2, u_3 contain the constant term, and terms of the first, second, and third degree respectively. If we take a certain point on the surface as origin we may not only find $u_0 = 0$ but $u_1 = 0$, all the terms in these functions disappearing.

In this case the tangent plane at the origin is indeterminate, and a cone $u_2 = 0$ can be drawn every generating line of which will meet the surface in three coincident points.

When $u_2 = 0$ represents a proper cone the point is called a conic node C_2.

When $u_2 = 0$ represents two planes we obtain a biplanar node, called a binode B_3, provided the planes do not intersect on the surface. When the two planes $u_2 = 0$ intersect upon the surface the biplanar node is called B_4, for the class of the surface is diminished by four. [Salmon's *Geometry of Three Dimensions*, p. 489.]

The intersection of the biplanes is called an *edge*. When the surface is touched along the edge by a plane we obtain a binode B_5. When the edge is *oscular* the binode is B_6.

To quote a note given by Dr Salmon: "In general, if a surface is touched along a right line by a plane, the right line counts twice as part of the complete intersection of the surface by the plane, the remaining intersection being of the order $n-2$. The line may, however, count three times, the remaining intersection being only of the order $n-3$. Professor Cayley calls the line *torsal* in the first case, *oscular* in the second. He calls it *scrolar* if the surface merely contain the right line, in which case there is ordinarily a different tangent plane, at each point of the line."

To apply this rule to surfaces of the third degree to distinguish between the binodes B_4, B_5 and B_6, we see that triple tangent planes pass through the edge in the case of B_5 or B_6; in the first of these, B_5, it is possible to take a tangent plane so that two of the straight lines in the section coincide, and one straight line remains distinct from the edge; in the second case, B_6, there is a tangent plane in which all three straight lines of the section move up to and coincide with the edge.

When the biplanes coincide the node is said to be uniplanar; or more shortly a Unode; a description of these is given on page 8.

When we wish to find how many singularities may exist on a surface we note that any singularity reduces the class of the surface.

For example, how many conic nodes can we have on a cubic surface? We find that a conic node reduces the class of a surface by two. Four conic nodes reduce it by eight. Therefore a cubic surface having four conic nodes is of the fourth class. If we attempt to express the equation to a cubic surface of the second class we find it impossible, for it no longer remains a proper cubic surface. Dr Salmon shews how this is done in the above-mentioned Chapter XI. The class of a surface is equal to the degree of its reciprocal. We treat of singularities by properties of the reciprocal surface. We say then that a cubic surface has not more than four conic nodes.

The equations indicated in the first chapter give all possible combinations of singularities.

C_2 reduces the class of the surface by two; B_3 by three; B_4 by four; the suffix shewing the reduction.

CHAPTER I.

EQUATIONS TO CUBIC SURFACES.

In 1849, as we read in Salmon's *Geometry of Three Dimensions*, Dr Cayley mentioned to Dr Salmon the fact that there were twenty-seven straight lines on a cubic surface.

Soon after this time Professor Schläfli investigated the properties of these lines, and papers published by him, together with Memoirs of Dr Cayley, continue to be the standard works upon the subject, as far as analytical methods are concerned.

Clebsch, Klein, and Zeuthen have examined the form of different surfaces, and Reye shewed how projective Geometry can be applied to surfaces of the third degree.

To commence then with Dr Schläfli's methods (indicated also in Frost's *Solid Geometry*), we take u, v, w, x, y, z as functions of the coordinates used, in the first degree, that is, if equated to zero they represent planes. Then it is known that these functions are subject to the condition $\mathsf{A}u + \mathsf{B}v + \mathsf{C}w + \mathsf{D}x + \mathsf{E}y + \mathsf{F}z = 0$ where the coefficients A, B, C, D, E, F are functions of two arbitrary constants. Now if any equation of the third degree be given it can be made coincident with

$$uvw + xyz = 0 \quad \text{.........................(1)},$$

subject to the added arbitrary condition that

$$\mathsf{ABC} = \mathsf{DEF} \quad \text{.........................(2)}.$$

Equation (2) gives a cubic condition between the above-mentioned arbitrary constants, and this cubic must have at least one real root, but may have three: these cases Schläfli tabulates as (...A) and (...B). From these equations we get the following cases:

Case I (A). u, v, w, x, y, z real. 27 real straight lines and 45 real triple tangent planes.

Case I (B). 15 real straight lines and 15 planes.

Case II (A). y and z imaginary and conjugate. 3 real straight lines and 13 planes.

Case II (B). 3 straight lines and 7 planes.

Case III (A). y, z, v, w imaginary, y conjugate to z, and v to w. 7 straight lines and 5 planes.

Case III (B). 3 straight lines and 7 planes.

Besides these cases we have cubic cones and cylinders, also zero values for u, v, w, x, y, or z, when the equation reduces to planes and surfaces of the second degree.

Further, the values must not be coincident, otherwise we get nodes which are considered later.

If the reality of the straight lines only, that lie on the surface, is required, we may proceed more simply thus:

Take the equation $uvw = kxyz$.

There are 19 constants implied in this equation, and by equating coefficients we have sufficient equations, and no more, to make this expression identical with any given equation of the third degree. One factor on each side, as u and x, must be real, but v, w and y, z may be imaginary and conjugate [Salmon's *Geometry of Three Dimensions*, 4th ed. p. 498].

If u, v, w, x, y, z are all real factors, then we must have at least nine real straight lines on the surface, that is to say, the intersection of the planes $u = 0$, $x = 0$; $u = 0$, $y = 0$; $u = 0$, $z = 0$; &c. Denote the straight line in the planes $u = 0$ and $x = 0$ by $[ux]$.

Take three of the straight lines that do not intersect, as $[ux]$, $[vy]$, and $[wz]$, and take the equation of any straight line meeting them. The constants involved in this last equation can be made to depend upon one variable, p suppose, for it is the generator of a known hyperboloid.

Express the condition that any point on the line is upon $uvw = kxyz$.

We find a cubic equation for p, and this must have one real root (Case I, B), or may have three (Case I, A). Now if we have three real roots we must have three real straight lines meeting $[ux]$, $[vy]$, and $[wx]$ that lie on the surface.

Similarly we have three straight lines meeting each of the other five sets of three non-intersecting lines. Also if we have three real roots for lines meeting the first set, we must have real roots for the other sets. This is proved by the consideration that we cannot have a plane section of the surface containing two real and one imaginary straight line.

Table of Reference.

[Numbers indicate straight lines. Three numbers in a bracket indicate three straight lines in a plane.]

(4, 6, 5), (13, 10, 3), (9, 8, 7), (4, 13, 9), (6, 10, 8), (5, 3, 7), (12, 25, 18), (24, 14, 17), (19, 16, 1), (12, 24, 19), (25, 14, 16), (18, 17, 1), (2, 21, 22), (20, 15, 27), (23, 26, 11), (2, 20, 23), (21, 15, 26), (22, 27, 11), (4, 12, 2), (13, 14, 15), (9, 1, 11), (5, 14, 11), (3, 1, 2), (7, 12, 15), (6, 15, 1), (10, 11, 12), (8, 2, 14), (4, 27, 16), (13, 23, 18), (9, 21, 24), (4, 26, 17), (13, 22, 19), (9, 20, 25), (5, 18, 21), (3, 24, 27), (7, 16, 23), (6, 22, 25), (10, 20, 17), (8, 26, 19), (6, 23, 24), (8, 27, 18), (10, 21, 16), (5, 19, 20), (3, 25, 26), (7, 17, 22).

N.B. To find whether two straight lines intersect, observe whether they occur in the same plane.

One side of every triangle intersects with one side of any other triangle.

The table of reference is taken from a paper by Mr H. M. Taylor [*Phil. Trans.* Vol. 185].

To connect this table with Dr Schläfli's equation we take the lines according to the following scheme, so that 9 is the intersection of $w=0$ and $x=0$, and 15 of $v=0$ and $z=0$, each line being the intersection of the plane vertically above it with that to the left of it.

	u	v	w
x	4,	13,	9,
y	2,	14,	8,
z	12,	15,	7.

If we take the lines forming a diagonal of this table, *e.g.* 4, 14, 7, we can find three straight lines on the surface that meet all of these, and no more.

By interchanging rows and columns we get in all six such diagonals. Therefore there are 18 beside the 9 at first found, making a total of 27.

We can take it that

5, 16, 17	meet	4, 14, 7.	
10, 18, 19	,,	13, 8, 12.	
1, 20, 21	,,	9, 2, 15.	
6, 26, 27	meet	4, 8, 15.	
3, 22, 23	,,	13, 7, 2.	
11, 24, 25	,,	9, 12, 14.	

Now we note that from the equation to the surface we know only the nine lines in the above scheme, and their intersections, also that there are sets of three lines each meeting three non-intersectors.

With regard to the intersections of the eighteen lines as given in the table of reference some proof is required. All these intersections can be obtained from the equation $uvw = kxyz$, but the necessary calculations are long and tedious.

Let us assume then that the intersections of 4, 13, 9, 2, 14, 8, 12, 15, and 7 are known; also that there are three straight lines 5, 16, 17 that meet 4, 14, 7 and lie on the surface, and three *only*, and three as 6, 26, 27 that meet 4, 8, 15, and so on for the other sets, but nothing is yet assumed as to the intersections of 5, 16, 17; 6, 26, 27; &c.

Draw a plane through the straight lines 4 and 5. This plane meets the straight lines 8 and 15 in points which we will call P and Q. The straight line PQ meets the surface in four points, for it cuts 4, 5, 8, 15. Therefore PQ lies on the surface.

Similarly if we draw planes through 4, 16, and 4, 17, we find straight lines that meet the three non-intersectors 4, 8, 15.

Thus we have found three non-intersecting lines that lie on the surface and meet 4, 8, 15.

But there are three, and only three, such straight lines that lie on the surface, namely, 6, 26 and 27, therefore the three lines found by the above construction must be the lines 6, 26 and 27. We also see that PQ meets 5; similarly the other two lines meet 16 and 17 respectively.

We, therefore, can state that the three lines 6, 26, 27 each meet one of the lines 5, 16, 17. Since we may take any other sets in the same way, it is evident that if we take any two sets of three non-intersectors as 4, 14, 7 and 4, 8, 15 having a common line, 4, then their three respective non-intersectors 5, 16, 17 and 6, 26, 27 intersect two and two as 5, 6; 16, 27; 17, 26.

If we apply this rule to each set in turn we find all necessary points of intersection.

We observe that there are five planes passing through the line 4, namely, 4, 2, 12; 4, 13, 9; 4, 5, 6; 4, 16, 27; and 4, 17, 26.

Dr Salmon uses the following method:

Take Dr Schläfli's result so far as to assert that there are twenty-seven straight lines on the surface, and that not more nor less than three of these may lie in one plane.

Draw any plane $u = mx$ through the intersection of the planes $u = 0$ and $x = 0$, the section through this line is completed by the intersection of $u = mx$ with the quadric surface $mvw = kyz$. If we express the condition that this section should become two straight lines m is involved in the fifth degree, therefore there are five and only five sections through a straight line on the surface in which the conic that completes the section becomes two straight lines. Therefore the problem reduces to this: arrange twenty-seven straight lines in sets of three, so that no pair occurs twice in the same set, and ten points of intersection lie on each line, while five planes and only five pass through each line. Taking ten points of intersection for twenty-seven lines, we count each point twice, therefore there are 135 points of intersection.

Taking five planes about each line we count every plane three times, once for each side, therefore there are 45 planes.

Arranging the lines according to the table of reference, we find the arrangement complies with the required conditions, and no change can be made without infringing them.

Another more direct method is given later when considering "double sixes" on the surface.

If any two plane triangular sections be taken their sides intersect two and two in one straight line. This is necessary, for the six sides of the triangles meet the straight line that is the intersection of the planes in six points, and these must coincide by pairs, for a straight line can only meet the surface in three points. It may also be proved thus: Let $x=0$, $y=0$ be two triangular plane sections of a cubic surface that do not intersect upon the surface. The equation to the surface must be of the form $xyz=\mathsf{S}$ where S is of the third degree, for the equation to the surface does not vanish when $x=0$ and $y=0$, so that x, y are not factors on opposite sides of the equation. When $x=0$ and when $y=0$, S must resolve into three linear factors, therefore we infer that the equation may be written $xyz=kuvw$; the three straight lines $x=0$, $u=0$; $x=0$, $v=0$; $x=0$, $w=0$ intersect the three $y=0$, $u=0$; $y=0$, $v=0$; $y=0$, $w=0$ on the straight line $x=0$, $y=0$.

It is useful to note that four non-intersecting straight lines can only be intersected by two non-intersecting straight lines, for if we take two groups of three from these four lines, and through them construct two hyperboloids which are fully determined by these lines, then these hyperboloids can only have two other common generators which must, therefore, meet the four given straight lines. That there are two and only two is most easily seen by taking a plane section of both surfaces.

We find two conics meeting in four points.

Two of these points lie on two of the given straight lines, and the others must be points on the remaining lines of intersection of the hyperboloids.

The two intersecting straight lines may be imaginary, or coincident.

If we assume that when straight lines become imaginary, they must do so by pairs, we arrive at the conclusion that the number of real lines on the surface is 27, 15, 7, or 3.

We note that the pairs may be taken in two ways, either two imaginary lines meeting four non-intersecting straight lines, or in a plane section we may have two imaginary and conjugate lines together with a real line.

The lines numbered 16 to 27 constitute a double six. None of the even numbers nor of the odd numbers intersect one another. One of the even numbers, as 16, intersects all the odd numbers except one, that is 17.

Each intersecting pair, as 16, 19, lie in the same plane with, and complete a plane section of the surface with one of the remaining 15 straight lines.

If we take 16 as an imaginary straight line, and agree that where there is freedom of selection to keep the numbers 1 to 15 real, we find the double six 16 to 27 is imaginary. Thus 16, 19, 1 form a plane section; now 16 is imaginary, therefore 19 or 1 is imaginary and conjugate to 16. As arranged we select 19. In the same way 21 is imaginary, being conjugate to 16 in another section. It is easy to see that 16 causes 19, 21, 23, 25, 27 to be imaginary, and these in turn cause 18, 20, 22, 24, 26 to become imaginary: 17 is imaginary as conjugate to 18 or one of the other even numbers. There is now no single imaginary straight line left, and, therefore, no other line to be taken as its conjugate, unless for some other cause one of the remaining 15 becomes imaginary.

Double sixes may be formed in 36 different ways, but no set is independent of another. If one double six is removed we cannot make a double six out of the remaining straight lines. In taking one double six we take four lines that go to make another. Suppose then two double sixes to become imaginary, we have seven real straight lines left. Next if three double sixes become imaginary 24 straight lines disappear and we have but three left. Therefore we have 27, 15, 7, or 3 real straight lines.

We next have to consider the coincidence of lines and the singularities produced in the surface.

At an ordinary node the tangent plane is replaced by a quadric cone [Salmon's *Geometry of Three Dimensions*, 4th ed. p. 488], called by Cayley C_2.

The quadric cone may degenerate into a pair of planes intersecting in a straight line called *an edge*. If the edge is not a line on the surface we have a binode B_3. If the edge is a line on the surface not torsal or oscular the binode is B_4. Next in the case of the binode B_5 the surface is touched along the edge by a plane, which however contains another straight line on the surface not coinciding with the edge, called a transversal. The edge is now said to be *torsal*. If the transversal moves up to the edge, and coincides with it, the edge is *oscular* and the binode is B_6. U_6, U_7, U_8 are uniplanar nodes where the quadric cone becomes a coincident plane-pair.

In U_6 the rays are three distinct lines, in U_7 two, and in U_8 all three coincide.

A straight line through a node being the coincidence of two or more straight lines on the surface is called a *ray*.

In the following equations where tetrahedral coordinates are used the equations to the faces are represented by $X = 0$, $Y = 0$, $Z = 0$, $W = 0$.

Where straight lines are represented by numerals they are taken from the Table of Reference.

One Conic Node C_2.

The equation to the surface may be written $XYZ = KWS$, where $S=0$ is the equation to a cone of the second degree having its vertex at $X=0$, $Y=0$, $Z=0$, that is, S is a homogeneous equation in X, Y, Z of the second degree and K a constant [Salmon's *Geometry of Three Dimensions*, p. 488, 4th ed.].

In Dr Schläfli's paper the kind and number of nodes on the surface is fully discussed. Taking the notation of page 1, we find that in a double six the lines have coincided by pairs to form six nodal rays, passing through the node. These represent 12 lines on the surface, any pair lie in the same plane with one of the remaining straight lines on the surface.

We derive from Dr Schläfli's cases of the reality and unreality of lines and planes on page 1 that we may have six real nodal rays and 15 real straight lines (Case I, A); next 12 lines may become imaginary, eight of which are included in the nodal rays; in other words, we have 4 real nodal rays, and 7 mere lines forming three plane sections of the surface. (Case I, B.)

Case II (A) may resolve itself into 2 real nodal rays, and 3 mere lines.

Lastly, the nodal rays may become imaginary and conjugate by pairs, we then only find 3 mere lines real.

The equation may at once be found when the lines on the surface are given.

The position of the conic node rays is limited by the fact that they lie on a cone of the second degree. The equations of these lines are necessary and sufficient to determine $S=0$ upon which they lie.

$W=0$ is a plane containing three of the mere lines that form a triangle. The conic node rays meet the sides of this triangle two and two. $X=0$, $Y=0$, $Z=0$ are the planes each containing two rays, and a side of the triangle meeting each pair respectively.

There now remains the constant K only. This is found by substituting in the equation the coordinates of any convenient point on one of the remaining lines. Therefore the lines sufficient to determine the surface are the six nodal rays, three mere lines meeting them two and two, and one point on the surface not upon these given lines.

Table of Reference.

The lines 16, 17; 18, 19; ... 26, 27 coincide by pairs to form conic node rays $r_1, r_2, \ldots r_6$ passing through the node, we have the planes $r_1r_2 1$, $r_3r_4 2$, $r_5r_6 3$, $r_1r_6 4$, $r_2r_3 5$, $r_4r_5 6$, $r_1r_4 7$, $r_2r_6 8$, $r_3r_5 9$, $r_1r_3 10$, $r_4r_6 11$, $r_2r_5 12$, $r_2r_4 13$, $r_1r_5 14$, $r_3r_6 15$. Planes are formed by the lines 1 to 15 as before.

ONE BINODE B_3.

The equation $S=0$ to the cone in C_2 resolves into two planes, and we may take as equation to the surface

$$XYZ = KW(X + bY + cZ)(X + b_1Y + c_1Z).$$

Comparing with Dr Schläfli's equations (page 1) to investigate the reality of the straight lines on the surface, we find (1) all the lines may be real; (2) both biplanes real, one containing a real ray and two imaginary ones, and the other three real rays; there are also three mere lines; (3) two real biplanes, each containing one real ray, and two imaginary and conjugate, and one mere line; (4) the two nodal planes imaginary and conjugate, we then have three mere lines remaining.

Counting each ray as three coincident lines we see that (1) gives 27 lines, (2) 15 lines, (3) 7 lines, and (4) 3 lines.

A plane turning about its edge cuts the surface in a curve with a cusp, which changes its direction to the opposite one whenever the turning plane has passed one of the two real nodal planes.

Table of Reference.

Let the conic node cone degenerate into two planes, but let the planes not intersect on the surface. Then three rays of the conic node as $r_1r_2r_3$ lie in one plane and $r_4r_5r_6$ also lie in one plane, the triangle $r_1r_2 1$ coincides with $r_1r_2r_3$, in other words 1 coincides with r_3. Similarly 5, 10, 3, 11, and 6 coincide with r_1, r_2, r_4, r_5 and r_6. The remaining triangles formed by rays with mere lines, and by the mere lines among themselves, are the same as those given for the conic node.

THE BINODE B_4.

The difference between this case and B_3 is that the intersection of the biplanes, or edge, lies on the surface. It represents also the coincidence of two conic nodes C_2 in considering the form of the surface.

The equation to the surface may be expressed as

$$WXZ + (X + Z)(Y^2 - aX^2 - bZ^2) = 0.$$

The lines on the surface are the edge of 6 coincident lines, four rays, two in each biplane of 4 coincident lines each, the transversal in the plane touching along the edge, and four mere lines.

(1) All these lines may be real, (2) the axis may be real, and two rays only in one nodal plane, and the transversal, (3) the axis and transversal only may be real, the rays being conjugate by pairs, (4) the axis may be real and the transversal and one plane through the transversal. In case (4) the axis counts as 4 real and 2 imaginary lines. We find then 27, 15, or 7 real straight lines.

The above equation to the surface is quoted direct from Dr Cayley's Memoir, as are also those for B_5, B_6, U_6, U_7 and U_8. He implies constant multipliers, which I have inserted for convenience in discussing some of the other equations. For example the above equation could have been written

$$KWXZ + (X + Z)(Y^2 - aX^2 - bZ^2) = 0.$$

Table of Reference.

The table may be found from that of B_3 by supposing two rays as r_3 and r_6 to move up to and coincide with one another: in this case 8, 4, 9, and 2 must move up to and coincide with r_1, r_2, r_4, and r_5. It may also be found from the case of two conic nodes, by supposing the nodes to coincide.

It is more symmetrical to take A, B, C, D the rays; E the edge; a, b, c, d the mere lines, and t the transversal. The planes are EAD, EBC, ABc, ACd, aBD, bCD, tbc, tad, Et.

The section Et is not triangular but is supposed to consist of two coincident lines crossed by one line; a diagram is given in the chapter of sections of a cubic surface, figure 5, section H, Chapter VI. We see also from this figure why t is called the transversal, or line crossing the edge.

THE BINODE B_5.

The equation to the surface is

$$WXZ + Y^2Z + YX^2 - Z^3 = 0.$$

In all previous cases, the equation to the surface can be found when the straight lines upon it are known, but though this surface may be found by taking the general case, and supposing lines to move up to and coincide with one another, we cannot find its equation in this way. This fact is noted by Cayley in his Memoir on Cubic Surfaces, *Collected Papers*, Vol. VI. p. 411. We find it by taking the case of B_3 and making one of the planes osculate along the nodal edge.

The planes are

$$Z = 0, \quad X = 0, \quad Y + Z = 0, \quad Y - Z = 0.$$

The lines are

$$X = 0, \quad Z = 0; \quad Y = 0, \quad Z = 0; \quad X = 0, \quad Y + Z = 0;$$
$$X = 0, \quad Y - Z = 0; \quad X - W = 0, \quad Y + Z = 0;$$
$$X + W = 0, \quad Y - Z = 0.$$

The edge is equivalent to 10 lines, the ray of the torsal plane to 5 lines, two rays of the ordinary biplane to 5 lines each, and there are two mere lines.

(1) All the lines may be real or (2) we may have two simple lines imaginary and two rays in the nodal plane, equivalent to 27 and 15 real straight lines respectively.

Dr Salmon remarks that B_5 may be considered as resulting from the union of C_2 and B_3; he shews how the above equation may be obtained [Salmon's *Geometry of Three Dimensions*, 4th ed. p. 489].

Table of Reference.

Denote the edge by E; the rays by A, B, C; the mere lines by S and T; the planes are EA, EBC, ACS, and ABT.

The Binode B_6.

The equation to the surface is

$$WXZ + Y^2Z + X^3 - Z^3 = 0.$$

The planes are $X = 0$ and $Z = 0$. The lines are the edge $X = 0$, $Z = 0$; equivalent to 15 lines and two nodal rays $X = 0$, $Y + Z = 0$; $X = 0$, $Y - Z = 0$, each equivalent to 6 lines. We have two species, (1) the nodal rays may be real or (2) conjugate.

Table of Reference.

In the table for B_5 one ray moves up to and coincides with the edge, the mere lines coincide with the rays which they respectively intersect. The planes are E by itself, and EBC.

The Unode U_6.

The biplanes in B_3 may coincide, we then have

$$W(X + Y + Z)^2 + XYZ = 0.$$

The planes are

$$X + Y + Z = 0, \quad X = 0, \quad Y = 0, \quad Z = 0, \quad W = 0.$$

The lines are the intersections of the planes $W=0$ and $X+Y+Z=0$ with $X=0$, $Y=0$, and $Z=0$ respectively. We find then three rays of 8 coincident lines, and three mere lines, which makes a total of 27. There are two species, for Y, Z may be real or imaginary and conjugate.

The Unode U_7.

Two rays in U_6 coincide, and therefore two mere lines coincide with the other ray.

The equation to the surface is

$$WX^2+XZ^2+Y^2Z=0.$$

The uniplane is $X=0$, the plane touching along the single ray is $Z=0$. The lines are $X=0$, $Y=0$, the torsal ray of 16 lines; $X=0$, $Z=0$, the single ray of 10 lines; and $Z=0$, $W=0$, the mere line.

The Unode U_8.

The equation of the surface is

$$X^2W+XZ^2+Y^3=0.$$

All the planes have coincided to one plane $X=0$, and all the 27 lines to one ray $X=0$, $Y=0$.

It is found by making the three rays in U_6 coincide.

Two Conic Nodes C_2.

The equation to the surface may be given as

$$WXZ+Y^2(aZ+bW)+KS=0,$$

where S is a homogeneous function of X and Y of the third degree.

The line joining the conic nodes is torsal, it is called the conic node *axis*, it is met by one mere line or transversal.

With regard to reality of lines (page 1) we have three cases (1) all lines real, (2) two conjugate planes and two real ones through the axis, and through the transversal one real and two conjugate; (3) planes through the axis conjugate by pairs, the transversal plane only is real. The axis represents four coincident lines, and each of the remaining conic node rays two; the mere lines are the transversal that cuts the axis, and six lines the intersections of biradial planes of one node with those of the other. Two conjugate planes through the axis contain four rays or eight lines, and two planes through the transversal, four more; therefore in case (2) the number of real lines is diminished by 12, leaving 15; the third case (3) leaves only three real lines, for the axis counts as two real lines and two imaginary, and there is also the transversal.

Table of Reference.

In the table for one conic node let 4, 14; 12, 8; 9, 15; and 6, 11 coincide by pairs to form rays R_1, R_2, R_3, R_4. Then the rays $r_5 r_6$ will coincide to give an axis A, joining two conic nodes, the second node being at the intersection of R_1, R_2, R_3, and R_4, which is also the coincidence of such points of intersection as 4, 6; 11, 14; &c.

Now beside the planes given by the remaining mere lines, which are the same as those given in the first table, we have planes $r_2 r_4 13$, $R_1 R_3 13, \ldots$ formed by pairs of rays from the same node with mere lines, two for each remaining mere line other than the transversal; next we have planes $Ar_1 R_1$, $Ar_2 R_2 \ldots$ formed by the axis with a pair of rays one from each node, and lastly At the section containing the axis and transversal, which is the line 3.

Any of these planes can be determined from the original table by noting which of the original lines are contained in each ray.

Three Conic Nodes $3C_2$.

The equation to the surface is

$$Y^3 + Y^2 (X + Z + W) + 4aXZW = 0.$$

With regard to the reality of the lines on the surface, using Dr Schläfli's method (p. 1), we get five species: (1) all the lines real, (2) the three axes real and three transversals, (3) two nodes conjugate and the nodal cone at the real node imaginary, (4) the same as the last case except that the nodal cone is real, and (5) a real axis joining two conjugate nodes, two rays of the real node, and the transversal of the axis real.

Comparing the values of the lines with the table given below, we find these cases represent 27, 15, 3, 7, and 7 lines respectively. A real axis joining two conjugate nodes counting as 2 real and 2 imaginary lines, in case (5).

Table of Reference.

Two of the conic node rays from each node, where there are two conic nodes, move up to and coincide with one another to form two axes, joining the two nodes to the third node. We may call the nodes A_2A_3, A_3A_1, and A_1A_2 where A_1, A_2, A_3 are the axes. There are six rays, two from each node, which we may call R_1, r_1 from A_2A_3, R_2, r_2 from A_1A_3 and R_3, r_3 from A_1A_2. There remain only three transversals t_1, t_2, t_3, which are mere lines, the axes each counting as 4 and the rays as two mere lines respectively.

The planes are $t_1t_2t_3$, $A_1A_2A_3$, $r_2r_3A_1$, $R_2R_3A_1$, $r_3R_1A_2$, $R_3r_1A_2$, $r_1r_2A_3$, $R_1R_2A_3$, $t_1r_1R_1$, $t_2r_2R_2$, $t_3r_3R_3$, A_1t_1, A_2t_2, and A_3t_3.

FOUR CONIC NODES C_2.

The equation to the surface is

$$AYZW + BXZW + CXYW + DXYZ = 0.$$

The equation to the plane containing the transversals is

$$AX + BY + CZ + DW = 0.$$

We may get (1) all the nodes real, (2) two nodes real and two conjugate, or (3) lastly four nodes conjugate two and two.

Taking the case of three conic nodes we find that rays through the nodes have coincided by pairs to form axes joining the fourth node to the other three; we therefore have six axes, the edges of the tetrahedron of reference, each representing four straight lines, and the three transversals.

When two nodes are real we have one real axis and the transversal or 7 real lines, when no nodes are real we have only the transversals.

Klein has shewn that the form of any cubic surface may be derived from one having four conic nodes.

Further, if we take a point on the surface, and from it draw a tangent cone, the quartic curve projected by this cone upon a plane reduces to two conics, their four points of intersection being the projections of the four nodes.

The straight lines on the surface are sufficient to determine it completely, and these in turn are fixed by the position of the transversal plane, the axes being known. This is evident from the equations given above, but may also be shewn geometrically.

First then to find the transversals, their plane being given. The plane cuts the axes in six points, which may be divided into three sets of two, taking those belonging to opposite edges together. Join each pair of points by straight lines; these lines are the transversals.

Next take any plane through one of the axes. We know that the section of the surface on this plane is made up of the axis and a conic section. But we clearly have five points on this conic already known, namely the two nodes on the axis, two points where the transversals that do not meet the axis cut the plane, and another where the axis representing the opposite edge of the tetrahedron cuts it.

By drawing planes through the axes we can thus find any number of sections of the surface, for five points are sufficient to fix the conic.

Table of Reference.

The planes are now reduced to the four faces of the tetrahedron made by the nodes, a triangle made by transversals, and six planes of the form At, in which lie an axis with the transversal that meets it.

[*In any plane section parallel to the plane of the transversals not only the asymptotes, but also the tangents at the points in which the axes meet it are parallel to the transversals, each tangent to the transversal corresponding to the axis. Six points with their tangents, and the directions of the asymptotes, are sufficient to determine the section.*]

Two Binodes B_3.

The equation to the surface is $WXZ + KS = 0$ where S is a homogeneous function of X and Y of the third degree.

The singular osculating plane and one at least of the other planes through the axis must be real, but W, Z may be imaginary and conjugate, so also may two of the three factors of S. From this double reason of partition we get four species [Dr Schläfli].

If the three factors of S are $aX - Y$, $bX - Y$, $cX - Y$, the lines on the surface with their equations are $X = 0$, $Y = 0$, an axis of 9 lines; and $aX - Y = 0$, $bX - Y = 0$, $cX - Y = 0$ taken in turn with $Z = 0$ and $W = 0$, these represent 6 rays of 3 lines each. The reality of the axis and two rays implies 15 real lines.

Table of Reference.

In the case of two conic nodes we have to suppose a double change, first the conic node rays which may be regarded each as two mere lines take up one mere line to become a binode ray, the axis takes up two more mere lines, and then one of the rays coincides with the axis. These changes give the system. We have the axis A, three binode rays b_1, b_2, b_3, and three B_1, B_2, B_3. The planes are $b_1 b_2 b_3$, $Ab_1 B_1$, $Ab_2 B_2$, $Ab_3 B_3$, $B_1 B_2 B_3$, and a plane containing the axis A only.

Three Binodes B_3.

The equation to this surface is $WXZ + Y^3 = 0$.

We have three axes of 9 lines each joining the three nodes which are angular points of the tetrahedron. A model of this surface in the South Kensington Museum will be described in a later chapter. We have two species, for X, Z may be real, or imaginary and conjugate.

There is no reason to quote the table of reference, for all the triangular sections have coincided with the biplanes. The planes are reduced to four, namely the common biplane $Y=0$, and the planes one for each binode $X=0$, $W=0$, $Z=0$.

The Binode B_3 + the Conic Node C_2.

The equation to the surface is

$$WXZ + KY^2Z + m(aX - Y)(bX - Y)(cX - Y) = 0.$$

There are two species representing respectively 27 and 15 real lines, the first when all the lines are real, and the second when the axis, two rays of the binode, one of the conic node, and one mere line are real.

Table of Reference.

In the table for B_3 let two rays r_1 and r_2 unite to form an axis A of 6 lines, we will call r_3 the binode ray B, and r_4, r_5, r_6 binode rays b_1, b_2, b_3. Of the nine mere lines six unite by pairs to form conic node rays c_1, c_2, c_3. There still remain three mere lines l_1, l_2, and l_3.

The planes are $b_1b_2b_3$, AB, $\mathrm{A}b_1c_1$, $\mathrm{A}b_2c_2$, $\mathrm{A}b_3c_3$, $\mathrm{B}l_1b_1$, $\mathrm{B}l_2b_2$, $\mathrm{B}l_3b_3$, $c_2c_3l_1$, $c_3c_1l_2$, $c_1c_2l_3$.

The Binode B_4 + the Conic Node C_2.

The equation to the surface is

$$WXZ + (X + Z)(Y^2 - X^2) = 0.$$

We cannot find the equation of the surface from any consideration of the position of the lines by geometrical reasoning. We need not therefore quote the table of reference. The equations to the straight lines on the surface are $X=0$, $Y=0$, the axis through the nodes counting as 8 lines; $X=0$, $Z=0$, the edge of the binode being a transversal, and counting as 6 lines; $X-Y=0$,

$Z=0$, and $X+Y=0$, $Z=0$, two biplanar rays of four lines each in the non-axial biplane; $X-Y=0$, $W=0$ and $X+Y=0$, $W=0$, two conic node rays of 2 lines each; and lastly $X+Z=0$, $W=0$, a mere line.

Dr Schläfli gives the equation as

$$WXZ+(X+Z)(Y^2-aX^2)=0,$$

and hence deduces two species according to whether a is positive or negative. For if a is negative the factors of $Y-aX^2$ are imaginary and conjugate. It is clear in this case that the binode and conic node rays become imaginary; this accounts for 12 lines and leaves us with 15 real lines.

The Binode B_5 + the Conic Node C_2.

The equation to the surface is

$$WXZ+Y^2Z+YX^2=0.$$

The planes are $Z=0$, $X=0$, $Y=0$, and the lines are $X=0$, $Y=0$, an axis of 10 lines; $X=0$, $Z=0$, an edge of 10 lines; $Z=0$, $Y=0$, a biplanar ray in the torsal plane of 5 lines; $W=0$, $Y=0$, a conic node ray of 2 lines.

There is but one species.

The Binode B_6 + the Conic Node C_2.

The equation to the surface is

$$WXZ+Y^2Z+X^3=0.$$

$Z=0$ is an oscular biplane; $X=0$ is an ordinary biplane; and the lines are $X=0$, $Y=0$, an axis of 12 lines; and $X=0$, $Z=0$, an edge of 15 lines.

There is but one species.

The Binode B_3 + Two Conic Nodes C_2.

The equation to the surface is $WXZ+Y^2(X+Y+Z)=0$. The lines are two axes $X=0$, $Y=0$, and $Z=0$, $Y=0$ of 6 lines each; an axis joining the conic nodes $Y=0$, $W=0$, of 4 lines; $X=0$,

$Y+Z=0$ and $Z=0$, $Y+X=0$, biplanar rays one in each biplane, being each a transversal of 3 lines; $W=Y=-Z$, and $W=Y=-X$, two conic node rays one through each node of 2 lines; and lastly the transversal $W=0$, $X+Y+Z=0$. We have two species, for X, Z may be imaginary and conjugate. In the second case the axis and the transversal are real, that is two real lines in the axis, two being imaginary, and one mere line $=3$.

Two Binodes B_3 + One Conic Node C_2.

The equation to the surface is

$$WXZ + XY^2 + Y^3 = 0.$$

The lines on the surface are an axis joining the two binodes $X=0$, $Y=0$ of 9 lines; two axes joining the binode to the conic nodes of 6 lines $Y=0$, $Z=0$, and $Y=0$, $W=0$; two biplanar rays one for each binode $Z=0$, $X+Y=0$, and $W=0$, $X+Y=0$ of 3 lines each.

We have three species, (1) all the lines real, (2) W and Z may be imaginary and conjugate, then the two biplanes are conjugate, the cone at the node is imaginary, but the point real, (3) W and $-Z$ may be imaginary and conjugate, then the two biplanes are conjugate and the node real.

One Binode B_4 + Two Conic Nodes C_2.

The equation to the surface is

$$WXZ + Y^2(X+Z) = 0.$$

The lines on the surface are two axes each of 8 lines $Y=0$, $X=0$, and $Y=0$, $Z=0$; one axis of 4 lines the conic nodes $Y=0$, $W=0$; the edge of the binode of 6 lines $X=0$, $Z=0$; and $W=0$, $X+Z=0$ a mere line. We have two species, for X and Z may be real or imaginary and conjugate.

Dr Schläfli gives a geometrical construction for the surface in the second case. Let a variable circle move having its diameter parallel to the axis of a fixed parabola, and intercepted between the curve and its tangent at the vertex, while the plane of the circle is perpendicular to that of the parabola, the circle generates the surface.

Cases numbered XXII and XXIII are scrolls having an infinite number of straight lines upon them. Dr Salmon gives a description of them in his *Geometry of Three Dimensions.*

It is not necessary to do more than give their equations, namely, $X^2W + Y^2Z = 0$ and $X(XW + YZ) + Y^3 = 0$.

Provided the equation does not resolve itself into factors so as to give three planes, or a plane together with a surface of the second degree, these twenty-three equations, together with the equations to cubic cones and cylinders, give all possible cases of equations to cubic surfaces.

Zeuthen has invented a very interesting method of finding the form of a cubic surface.

Take a point O upon the surface, but not upon one of the straight lines, call this point the "centre of projection."

Take also a fixed plane parallel to the tangent plane at O, as "plane of projection." Draw a straight line from O to meet the surface in P and Q, and the plane of projection in M. Let P be nearer to O than Q. When P and Q are both real points P traces out a sheet of the surface which we call the visible sheet, while that traced out by Q is invisible.

When P and Q are coincident M traces out a quartic curve.

If M be taken within the four closed curves of the quartic P and Q are both imaginary. One or more of these closed curves may be represented by infinite branches.

It may be stated then that the projection of any cubic surface from a point O upon it, on a plane of projection parallel to the tangent plane at O, is a quartic curve. [See an article by Zeuthen, *Math. Annalen*, Vol. VIII. p. 1.]

The projections of twenty-four of the twenty-seven straight lines are double tangents to the four closed curves of the quartic, each touching two of the four.

If a straight line may be considered during part of its length as traced out by P on the visible sheet, when its projection touches the quartic, the part produced must be taken as the locus of Q, that is, it becomes invisible.

It may be concluded that there are four openings or holes on every cubic surface having twenty-seven real straight lines, and that twenty-four of these lines pass each through two openings.

A double six passes through each opening, and a double six between each opening and the next.

The surface can be deformed still keeping the same general outlines to one having four conic nodes, the quartic corresponding to this second surface having four nodes must reduce to two conics intersecting in four real points, and the twenty-four straight lines on the surface coincide by fours to form six conic node rays, which project into the six straight lines joining the intersections of the conics by pairs.

Now to apply this more conveniently Zeuthen proceeds as follows. Take the point O at a point where the curvature is elliptic, and take the tangents of the first kind, that is the four double tangents to the quartic each touching the same oval twice, as imaginary. Replace the ovals for convenience by circles, then the twenty-four double tangents each touching two ovals are represented by the twenty-four double tangents to the circles.

A double six touches each circle. If therefore one oval becomes imaginary a double six disappears. Similarly if two ovals become imaginary another double six, partly involved with the first, also disappears. When the third oval is imaginary it is clear that all twenty-four of the tangents have disappeared, so that as far as the lines are concerned we need not consider the fourth oval, but this does not prevent the fourth oval from existing as a contour of the surface.

Some very good figures and sketches of surfaces having one conic node, symmetrical in form, described by Klein, will be found on the last few pages of *Math. Annalen*, Vol. VI., 1873.

In 1894 lectures were given by Klein in America which were reported in the *Evanston Colloquium*. In one of these lectures

(p. 27), after reference to the cubic curves discovered by Sir Isaac Newton, Klein mentions the methods, just described, by which Zeuthen derives the shape of cubic surfaces from quartic curves.

Next he proceeds to describe how he first came to construct the models of cubic surfaces exhibited by him at the World's Fair in Chicago.

He tells his audience that he was encouraged to make this series of models by one constructed by Clebsch in 1872.

This particular model was symmetrical, and was that of a cubic surface having twenty-seven real straight lines. It was known as the diagonal surface.

Dr Salmon in his *Geometry of Three Dimensions* states that one of the most useful equations for investigating the properties of cubic surfaces is

$$ax^3 + by^3 + cz^3 + dv^3 + ew^3 = 0,$$

where x, y, z, v, w involve the current coordinates in the first degree, and their sum is identically zero. He shews that the equation is perfectly general if there is no multiple point, for the number of independent constants involved is nineteen. [*Geometry of Three Dimensions*, 4th ed. p. 491.]

Now suppose a, b, c, d, e to become equal, we then have Clebsch's diagonal surface. We may put a, b, c, d, e each equal to unity. The reason that it is called the diagonal surface is this: Suppose we take any one of the planes $x = 0$, the other four planes cut it in a quadrilateral figure. The three diagonals of this quadrilateral are straight lines on the surface.

In this way we get 15 straight lines on the surface, three in each of the five planes. That this is the case is evident from the equations, for if we take $x = 0$, employing the condition

$$x + y + z + v + w = 0,$$

we find that $$x^3 + y^3 + z^3 + v^3 + w^3 = 0$$

reduces to $$3(y + z)(z + v)(y + v) = 0,$$

which with $x = 0$ represents the above-mentioned diagonals.

It is clear that if we have nine straight lines, so related that if written as a determinant with three rows and three columns, every line intersects every line in the same row and every line in the same column, then a family of cubic surfaces can be described that pass through these nine lines; for we may take the equation to the surface as $xyz = \kappa uvw$ where κ is indeterminate. See page 4.

Now κ may be fixed by assuming any point we like, not on the nine straight lines, nor in a plane with any three that intersect, to be on the surface.

When κ is known we find 18 straight lines, 12 of which may be imaginary, that lie on the surface.

But we may have already selected the point so as to lie on one of these lines, and it is easy to see that we may do so, for every value of κ must give at least one straight line on the surface; conversely if we take the arbitrary point as on this straight line, we obtain but one value of κ, which must be the same as before.

Thus we arrive at the conclusion that the equation to the surface is fully determined by nine straight lines arranged as above that lie on the surface, and a tenth straight line on the surface, that meets three non-intersecting lines of the nine.

Further, this tenth straight line may be fixed by assuming that it meets one of the straight lines already known, at an arbitrary point: for example, if we take a point on 4 as that at which 16 meets it, then 16 is fixed, for only one straight line can be drawn through this point to meet 14 and 7. It is found by drawing a plane through the fixed point and 14 to meet 7 in a second point. Joining the fixed point to this second point we get but one position for 16.

Sturm commences his text-book on cubic surfaces by taking such a system of lines.

We ensure the reality of 15 straight lines on the surface, and these are sufficient to determine it completely, but 7 or 3 are not sufficient.

To secure 27 real straight lines it is best to place a double six in position, for example 16 to 27; the other 15 straight lines must be real.

For example, 1 is the intersection of the planes containing 19, 16 and 17, 18 which are real planes completely fixed in position, for each contains two known straight lines. By using the table of reference we can pick out pairs of planes for all the lines from 1 to 15.

To place a double six in position we may take any five non-intersecting straight lines provided they all meet the same straight line. Take 16, 18, 20, 22, 24 that all meet 27. Now we know that two straight lines can always be drawn to meet four non-intersecting straight lines; they may be imaginary, but if one is real the other must be so. We have given that 27 meets 16, 18, 20, 22. There is but one other straight line that meets them, and this we number 25. Selecting sets of four in this way from the five straight lines we get 17, 19, 21, 23. Now 26 meets these four lines, so that the double six is completely determined.

Also from analytical considerations these five lines 16, 18, 20, 22, 24 meeting 27 are sufficient to give the cubic surface, for the five straight lines limited by the above condition give nineteen equations to find nineteen constants.

Having found the numbers 16 to 27 we proceed as shewn above to find the lines numbered 1 to 15 as intersections of pairs of real planes.

If we bear in mind that any plane triangular section must meet any other one so that one side of one meets one side of another, and no two sides of one section meet one of another, also that two planes meet in one straight line, we can easily verify the table of reference.

For example, as above-mentioned, 1, 19, 16 and 1, 17, 18 are planes, so also are 3, 24, 27 and 3, 25, 26.

Now we know that 19, 24; 16, 27; and 17, 26; 18, 25 are respectively pairs of intersections, therefore 1 meets 3.

Before proceeding to describe Reye's geometrical methods it may be interesting to examine their analytical meaning.

Let $X_1 = 0$, $Y_1 = 0$, $Z_1 = 0$ be the equations to three planes passing through the point S_1, and let $X_2 = 0$, $Y_2 = 0$, $Z_2 = 0$; $X_3 = 0$, $Y_3 = 0$, $Z_3 = 0$ be the equations to planes passing respectively through S_2 and S_3.

We define
$$LX_1 + MY_1 + NZ_1 = 0$$
$$LX_2 + MY_2 + NZ_2 = 0$$
$$LX_3 + MY_3 + NZ_3 = 0$$

as corresponding planes drawn from centres S_1, S_2, S_3. Eliminating L, M, N we find that the equation

$$\begin{vmatrix} X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \\ X_3 & Y_3 & Z_3 \end{vmatrix} = 0$$

represents a cubic surface.

Corresponding rays from S_1, S_2, S_3 are the intersection of corresponding planes.

We may state then that

$$lX_1 = mY_1 = nZ_1 \text{ and } lX_2 = mY_2 = nZ_2$$

are corresponding rays of the centres S_1 and S_2. Corresponding rays do not generally intersect, but when corresponding rays from two pencils intersect their points of intersection lie on a twisted cubic.

For we find $X_1 : X_2 :: Y_1 : Y_2 :: Z_1 : Z_2$ so that a point of intersection of corresponding rays lies upon both conicoids $X_1Y_2 = X_2Y_1$ and $Z_1Y_2 = Y_1Z_2$. Again the intersection of two such corresponding rays lies upon the cubic surface *independently of the third centre* S_3, for it causes two rows of the determinant representing the equation to the surface to become proportional. Therefore we see that every point on a certain twisted cubic, which we may call the twisted cubic S_1S_2, lies upon the cubic surface independently of the third centre S_3.

We note that if we take a fourth set of planes,

$$X_1 + mX_2 = 0, \quad Y_1 + mY_2 = 0, \quad Z_1 + mZ_2 = 0$$

passing through a point S_4 on the surface, then corresponding planes from S_4 may be used in place of those from S_1, S_2, or S_3 to generate the surface, because we may add to any row of a determinant the multiple of any other row.

Any secant or straight line cutting the twisted cubic S_1S_2 twice is the intersection of two corresponding planes from the centres S_1 and S_2. Therefore there is one plane of S_3 that corresponds to each secant, because it corresponds to both planes that by their intersection give the secant. Also one secant and only one can be drawn from any arbitrary point. It is clear then that the surface may be generated by the intersection of secants of S_1S_2 with corresponding planes from S_3.

Each secant therefore meets the surface three times, twice on the twisted cubic and once where the corresponding plane from S_3 meets it. The points at which any secant drawn from an arbitrary point meets the cubic may be real, coincident, or imaginary.

We may next proceed to shew that the surface may be generated by any three centres S_1, S_2, S_3 on the surface, not in one straight line. In the first place it is proved that any twisted cubic may be generated by pencils (or sets of corresponding rays) from any two centres upon it, the secants still remaining as intersections of corresponding planes.

We then take S_4 any point on the twisted cubic S_1S_2 and shew that any point on the twisted cubic S_4S_3 is on the surface.

It is assumed that S_4S_3 can by moving S_4 along S_1S_2 be made to pass through any point on the surface, in the same way that we can make any straight line through the vertex A of a triangle pass through any point P in the plane, by so selecting a point D in the base that ADP is a straight line.

It should here be stated that Reye establishes a complete correspondence between twisted cubics on the surface, and straight lines in a plane.

We see then that the surface may be generated by centres at any three points upon its surface not in one straight line. For we have shewn that the surface may be generated by the secants of a twisted cubic and corresponding planes from another centre, and that we can so alter the position of this twisted cubic as to make it pass through any point on the surface: further, the twisted cubic may be generated by any two centres upon it. It is evident then that by successive changes we may take any three selected centres.

It will be observed that we get another set of twisted cubics by taking

$$X_1 : Y_1 :: X_2 : Y_2 :: X_3 : Y_3.$$

The propositions which can be now established are evident from analytical considerations.

I. If four points lie in one straight line, that line lies entirely on the surface.

II. If a plane passes through a straight line on the surface the section is completed by a conic and *vice versâ*.

III. A plane section may degenerate into three straight lines.

IV. Two twisted cubics as S_1S_2, S_1S_3 have two and only two common secants not passing through S_1, that is the secants through S_1 not being included, unless the cubic surface degenerates into a plane and surface of the second degree. From this proposition we find the twenty-seven straight lines on the surface.

Many properties of plane cubic curves are derived from considering a curve as generated by a pencil of lines and a pencil of conics.

The method used may be illustrated as follows :

Take the equation of a cubic curve as

$$uvw = kxyz$$

where u, v, w, x, y, z equated to zero represent straight lines in the plane.

If we consider $muv = kxy$ as a pencil of conics, any given value of m fixing one of the system, a pencil of conics being a series described through four given points ; and take $w = mz$ as a corresponding pencil of straight lines drawn through $w = 0$, $z = 0$. Then to each conic corresponds a straight line, and each conic by its intersection with its corresponding straight line generates the curve $uvw = kxyz$.

CHAPTER II.

PROJECTIVE PENCILS.

On the Construction of Cubic Surfaces by Geometrical Methods.

A cubic surface is the locus of the intersection of three corresponding planes of three projective pencils, the centres of which are not in one straight line. [Reye.]

We have first to explain what is meant by projective pencils, and though the subject is fully discussed under the heading Geometry in the *Encyclopædia Britannica*, Vol. x., it will not be out of place here to give a brief description of Reye's methods.

First, then, take any number of points A, B, C, D, E, in a straight line and two points S and S′ in the same plane with it. Join SA, SB, SC, ... and S′A, S′B, S′C, ... Then we have two *plane perspective* pencils of lines or rays drawn from S and S′.

Turn one of these pencils through any angle, so that S′A, S′B, S′C, ... take up a new position as S′A′, S′B′, S′C′, ... Then SA, SB, SC, ... and S′A′, S′B′, S′C′, ... are not now perspective but *projective pencils.*

We observe that three rays in one pencil being made to correspond to three of another, the correspondence is uniquely determined.

For if SA, SB, SC, be given, and S′A′, S′B′, S′C′, and any other ray be taken as SD, then if we take the ray S′D′ so that the anharmonic range $[ABCD] = [A'B'C'D']$, we know that the ray $S'D'$ is uniquely determined, and similarly for any other ray.

It can easily be shewn in the same way by use of the principle of anharmonic ratio as given in most text-books on conic sections, that if two pencils are perspective or projective to a third, they are projective to one another.

It may also be assumed [*Geometrical Conics*, by Taylor, or Smith] that if we take four points K, L, M, N, and find that the anharmonic range $S[KLMN] = S'[KLMN]$, then the six points S, S′, K, L, M, N are on the same conic. This theorem may be used to prove that the corresponding rays of two *projective* pencils intersect on a conic.

For suppose corresponding rays of two projective pencils to intersect at K, L, M, their poles being S and S′. Describe a conic through K, L, M, S, S′, and let a ray from S meet the conic in N. Join S′N. Then since K, L, M, N, S, S′, are on the same conic, we find the anharmonic ratio $S[KLMN] = S'[KLMN]$, but as stated above this is a necessary and sufficient condition that the rays SN, S′N should correspond. In the same way we may proceed with any other ray, and as the correspondence is unique, and SN can meet the conic at no other point, there is no other solution of the theorem.

Next to consider axial pencils. Take two straight lines SP and S′P′, and through them draw planes to meet another given plane, if we suppose P and P′ for convenience to be in this given plane, and in it take any straight line having points A, B, C, D, ... upon it, then to each point A, B, C, D, ... correspond two planes SPA, S′P′A ; SPB, S′P′B ; &c. ... . Also to each pair of rays PA, P′A ; PB, P′B, correspond pairs of planes, passing through the axes SP and S′P′. If therefore in a plane we take two perspective pencils PA, PB, PC, ... , P′A, P′B, P′C, ... , and take any two points S and S′ not in the plane, and through SP and S′P′ called axes, we draw sets of planes to contain PA, PB, PC, ... , and P′A, P′B, P′C, ... , we get two axial pencils of planes.

Now any straight line drawn to meet two axial pencils that are perspective to one straight line is divided by their planes into two sets of points that are projective, for by using successive anharmonic ratios we may shew that each set of points is projective to the range A, B, C, D, ...

If one of these axial pencils is turned through any angle, it is clear that its anharmonic properties remain the same.

Therefore we may state that two axial pencils are projective if any plane section of one is projective to any plane section of the other.

Corresponding planes of two projective axial pencils not being perspective intersect in straight lines that generate a conicoid. This is evident from the fact that two plane projective pencils generate a conic. For two projective axial pencils are cut by a plane in two plane projective pencils, and corresponding rays of these plane pencils meet on a conic, and these corresponding rays lie in corresponding axial planes, so that the intersection of corresponding planes generates a conic in this plane section.

This being true for every plane section, it is clear that a conicoid is generated by the intersection of corresponding planes of two projective axial pencils.

We now come to projective pencils in space of three dimensions.

Take any two points S and S′ not in a given plane, and in this plane take any number of points A, B, C, D, ... Join SA, SB, SC, SD, ..., and S′A, S′B, S′C, S′D, ... We call these two perspective systems of rays, where SA, S′A; SB, S′B; SC, S′C; SD, S′D, are said respectively to correspond. Two planes that contain corresponding rays as SAB, S′AB are said to correspond.

Now let the pencil of rays centre S′ be turned into any other position, the rays still keeping the same position with regard to each other, then the rays S′A, S′B, S′C, S′D, ..., take up some new position as S′A′, S′B′, S′C′, S′D′, ... The pencils SA, SB, SC, SD, ..., S′A′, S′B′, S′C′, S′D′, ..., are projective pencils, where SA, SB, SC, SD, ..., are rays corresponding to S′A′, S′B′, S′C′, S′D′, ... As in perspective pencils corresponding planes contain corresponding rays, and two corresponding planes of one pencil intersect in a ray, that corresponds to the intersection of the two corresponding planes of the second pencil.

Pencils that are perspective to the same pencil are perspective or projective to one another.

These pencils may be supposed to be built up or composed of axial pencils. For take a plane containing points A, B, C, D, ..., and two centres S and S′ not in the plane, and take also two corresponding rays SP and S′P, then if we take any number of planes SPA, SPB, SPC, ..., these will correspond to S′PA, S′PB, S′PC, ..., in two axial pencils, the axes being SP and S′P.

In this case the pencils are perspective, but if one of them be turned through any angle, its projective properties remain the same, and the two pencils are so constructed that to every axial pencil in one corresponds an axial pencil in the other.

Corresponding rays of two projective pencils do not generally intersect, but when they do the locus of their points of intersection lies on a twisted cubic.

Let SS′ correspond to SA when considered a ray of the pencil centre S′ and to S′B′ when considered a ray of the other pencil. Then SA and SS′ may be taken as the axes of two projective axial pencils which, as we have shewn, generate a conicoid, and since the axes intersect this conicoid must be a cone, for every generator passes through S. Similarly a cone is generated by the axial pencils SS′ and S′B′, where SS′ is now taken as a ray of the first pencil. At every point at which these cones meet it will be found that corresponding rays of the pencils intersect at centres S and S′, for corresponding planes meet in corresponding rays.

When the cones have a common plane section, they intersect in a conic section, the twisted cubic in this case being made up of the conic and the straight line SS′. The conic cuts SS′, but is not in the same plane with it.

The correspondence of two pencils is uniquely determined when four rays or four planes of one correspond to four rays or four planes of the other.

The proof of this proposition follows from the fact shewn above that two projective plane pencils are determined by three rays of each. For take four rays SA, SB, SC, SD, to correspond to four rays S′A′, S′B′, S′C′, S′D′. Take two axial pencils having SA, S′A′ as axes meeting any given plane in A and A′. Then the plane pencils AB, AC, AD, and A′B′, A′C′, A′D′, are determined as projective to one another by three rays of each. Next taking axial pencils with axes SB and S′B′ we can find any point of the first pencil corresponding to any given point of the second in the given plane. For example, take Q any point in the given plane as lying on a ray SQ of the first pencil. Join AQ, BQ, then by taking axial pencils as shewn above with axes SA, S′A′; SB, S′B′, respectively, we find the pencils AB, AC, AD, AQ, and BA, BC, BD, BQ, and construct the corresponding pencils A′B′, A′C′, A′D′, A′Q′, and B′A′, B′C′, B′D′, B′Q′. Then A′Q′ and B′Q′ intersect in Q′, which is thus uniquely determined, and the ray S′Q′ will correspond with SQ.

It is clear that four planes equally determine correspondence, for corresponding planes intersect in corresponding rays.

If three points A, B, C, on a straight line correspond to themselves, then two ranges of points become perspective in this line. For if we have two anharmonic ranges on the same line, A, B, C, D, and A, B, C′, D′, so that [ABCD] = [ABC′D′], and if C′ moves up to C, D′ moves up to D, and as this is true for every position of D, the ranges become perspective.

Before proceeding to consider cubic surfaces it is necessary to prove a few of the properties of twisted cubics, namely :—

(i) A twisted cubic may be generated by two projective pencils, the centres of which are at any two points upon it.

(ii) Only one secant can be drawn to a twisted cubic from any point not upon the curve to meet it at two points which may be real, coincident, or imaginary.

(iii) Every secant is the intersection of two corresponding planes of the pencils.

If a_1 and a_2 be two corresponding rays in the pencils S and S′, then the corresponding planes in the axial pencils having a_1 and a_2 as axes generate a ruled quadric surface. If P be any point on the cubic, and p_1 and p_2 be the corresponding rays in S and S′ which meet in P, then to the plane $a_1\, p_1$ in S corresponds $a_2\, p_2$ in S′; these therefore meet in a line through P. This may be stated thus: those secants of a cubic which cut a ray a_1 drawn through the centre S of a pencil form a ruled quadric surface which passes through both centres and which contains the twisted cubic. Of such surfaces an infinite number exists. Every ray through S or S′ which is not a secant determines one of them. If however the rays a_1 and a_2 are secants meeting at A, then the ruled surface becomes a cone of the second order having A as centre. All secants which pass through a point on the twisted cubic form a cone of the second order, and the projection of a twisted cubic from any point on the curve to any plane is a conic. If a_1 is not a secant, but made to pass through any point Q in space, the ruled quadric surface determined by a_1 will pass through Q. There will therefore be one secant and only one passing through Q, for if two such lines pass through Q then the lines SQ and S′Q will be corresponding lines, hence Q will be a point on the cubic, and an infinite number of secants will pass through it. Hence through any point in space one and only one secant to the cubic can be drawn.

The fact that all the secants through a point on the cubic form a quadric cone shews that the centres of the projective pencils generating the cubic are not distinguished from any other points on the cubic.

If we take two points S and S′ on the cubic and draw the secants through each of them we obtain two quadric cones which have the line SS′ in common, and which intersect besides along the cubic. If we make these two pencils having S and S′ as centres projective by taking four rays on one cone as corresponding to the four rays on the other which meet the first on the cubic, the correspondence is determined. These two pencils will generate a cubic, and the two cones of secants having S and S' as centres will be identical with the above cones, for each has five rays in common with the first, namely, the line SS' and the four lines determined for the correspondence, therefore these two cones intersect in the original cubic. Therefore on a twisted cubic any two points may be taken as centres of projective pencils which generate the cubic, corresponding planes being those which meet on the same secant. Of the two projective pencils at S and S' we may keep the first fixed and move the centre of the other along the curve. The pencils will hereby remain projective, and a plane M in S will be cut by its corresponding plane M' always in the same secant a; while S' moves along the curve the plane M' will turn about a describing an axial pencil.

We note that a twisted cubic is a curve of the third order, for a plane cuts the two quadric cones which generate it in two conics which intersect in four points, and one of these is on the line SS' joining the centres of the pencils, while the other three are on the twisted cubic.

Now suppose we take three pencils having centres S, S', S'', that are projective, then we take as definition of a cubic surface, that it is the locus of the intersection of three corresponding planes of these pencils. It will be found necessary to make certain limitations to obtain a true cubic surface, beside the one already given, that the centres of the pencils are not in one straight line. For example, if the three pencils are perspective two and two, the surface obviously degenerates into three planes, and if two pencils are perspective we get a plane and a quadric surface.

It has already been shewn that the corresponding rays of two pencils that intersect meet on a twisted cubic. Let P be any point on the twisted cubic generated by S and S'. Join $S''P$, and at S take SQ to correspond to $S''P$ at S'', and $S'Q'$ to correspond to the same ray $S''P$, at the centre S'. Take $S''M$ at S'' to correspond to the ray SP at S, then we know it will also correspond to $S'P$ at S'. By definition corresponding planes are those containing corresponding pairs of rays, therefore QSP, Q′S′P, PS″M are three corresponding planes of the pencils S, S′, S″, and they meet at P, therefore P is a point on the surface. We may state then that the twisted cubic generated by the pencils S and S′ lies on the cubic surface independent of the position of S″. Similarly the twisted cubics joining the centres S′ and S″ and generated by the pencils having centres at S′ and S″ lie on the surface. Now it has been shewn that the secants of the twisted cubic joining S and S′ are the intersections of corresponding planes of the pencils at S and S′, therefore the point at which the corresponding plane at S″ meets a secant a, which is the intersection of the corresponding planes at S and S′, is the intersection of three corresponding planes at S, S′, and S″, and lies on the surface. Therefore this secant cuts the surface three times, namely, the two points at which it cuts the twisted cubic, and again where the corresponding plane from S″ meets it. The first pair of points may be imaginary. We note that it has been proved that the twisted cubic may be generated by two projective pencils at any two points S and S′ upon it.

Now take the three twisted cubics SS′, S′S″, and S″S, and upon one of these, S′S″, take a point X. [It will be convenient to take as figure a plane triangle SS′S″, and mark a point X upon its base.] Now take any point P upon the twisted cubic joining S and X generated by two projective pencils at S and X, the pencil at X being projective to those at S′ and S″, and therefore to S. P will lie upon the surface, for just as in the last proposition we can shew that it is the intersection of three corresponding planes from X, S, and S′, and also from X, S, and S″, and therefore of three corresponding planes from S, S′, and S″, the original pencils, therefore the point P lies on the surface.

By moving X along the twisted cubic S′S″, we can make the twisted cubic XS pass through any point P on the surface.

We find then that the surface may be generated by projective pencils at any three points upon the surface not in one straight line. It follows that no straight line generally can meet the surface at more than three points, for taking a straight line that meets the surface in P and Q, then PQ is a secant of a twisted cubic passing through P and Q. If the surface be generated by pencils at P and Q and any other point S on the surface, PQ can only meet the surface again where the corresponding plane from S meets it.

We say no straight line can generally meet the surface at more than three points, for suppose the three corresponding planes do not meet at a point, but in the same straight line, then every point of this straight line is on the surface.

It is clear also that if any straight line contains four points on the surface it lies entirely on the surface, for take the points P and Q as above as secant of a twisted cubic, through P and Q, let S be another point on the surface, and let the surface be generated by pencils having centres at P, Q, and S. Now there is but one plane at the centre S corresponding to the secant PQ, that is to the corresponding planes that determine the secant PQ. If therefore this plane meets the line PQ in two points it must contain it altogether, and therefore the line PQ lies entirely on the surface.

Next let SPQ be a straight line on the surface, and let the surface be generated by pencils at S, S′, and S″, where S′ and S″ lie in a plane through PQ but not upon PQ, S′ and S″ being upon the surface.

Now it is clear that the plane SS′S″ corresponds to itself in the pencils at S′ and S″, for corresponding planes from S′ and S″ meet in PQ. Therefore we have a number of rays of the pencil S′ being the intersections of other planes of the pencil S′, that all lie in the plane S′PQ, and correspond to rays that are the intersections of corresponding planes of the pencil S″, and also lie in the plane S′PQ; that is to say we have two plane projective pencils in the plane S′S″PQ, and the intersections of the corresponding rays of these pencils lie on the surface. But it is known that these pencils generate a conic. Therefore the plane section through any straight line is made up of this straight line and a conic section.

We next proceed to shew that a cubic surface can be described through any double six and is determined by it.

Take a double six U, V, W, X, Y, Z, and u, v, w, x, y, z, where any one of the first group as U meets any five of the second group as v, w, x, y, z, but does not meet any of the remaining lines.

Take three points S, S′, S″, as centres of pencils that generate the surface, and select them so that they lie on the lines x, y, z respectively, but not at points of intersection with other lines of the system. Let the projectivity of the pencils be determined by the lines W, X, Y, Z, that is to say let the planes WS, WS′, WS″ correspond, and so on for the other lines. Then it is clear that these lines are on the surface, for each is the intersection of three corresponding planes of the three pencils. u and v must lie on the surface, for they both meet W, X, Y, Z, and have each four points on the surface.

Next x, y, z are on the surface, for each passes through a centre and three other points on the surface, namely, where they meet three of the lines W, X, Y, Z.

U meets x, y, z, v; and V meets x, y, z, u; therefore both these lines are on the surface. Similarly w meets U, V, X, Y, Z, and is also on the surface.

There is no need to shew how to find the other fifteen straight lines on the surface, for this has been done before. We only use the same rule, namely, that when a straight line has four points common with the surface, it lies altogether upon it.

Suppose we take U, V, W, X, Y, that all meet the line z. It is known that the remaining lines of the double six can be found by continuing to draw pairs of lines to meet four non-intersecting lines, where one of the pair is already fixed. Therefore we conclude that five straight lines that meet one straight line are sufficient to determine a cubic surface.

The geometrical meaning of perspective and projective may be indefinitely extended. To repeat the simplest case, take a range of points A, B, C, D, ... in one straight line, and two points S and S_1 not in the line. The pencils SA, SB, SC, ..., and S_1A, S_1B, S_1C, ..., are perspective, and if one is turned about through any angle, the rays still keeping the same position with respect to one another, the pencils are now said to be projective.

By the principle of anharmonic ratio it may be proved that if we have any number of systems, any one of which is perspective to some other one, then, if not already perspective, every one of the system is projective to any other.

The definition is extended as follows. Two systems are said to be perspective when corresponding elements of one system lie upon or pass through corresponding elements of the other, each to each.

Two systems are projective when both are perspective each to one of a series of systems which are so related that any one of the systems is perspective to some one of the others.

To take an illustration. Take a system of conics passing through four points, and take some straight line having upon it a range of points A, B, C, D, ... , not including the four points of which no three are in one straight line. If we cause conics of the system to pass through A, B, C, D, ... , they are thereby a definitely fixed series. If we then join A, B, C, D, ... , to a point S, the pencil of rays thus formed is projective to the series of conics, and any other pencil that is projective to it is projective to the series of conics. For the range of points A, B, C, D, ... , are perspective to the pencils.

Again, take two circles having a common tangent at S. We find that any ray drawn through S has on it a point of one circle that corresponds to a point on the other. The two series of points on these circles form two projective sets of points, each being perspective to the same pencil of rays from S.

These circles may be projected into conics having a common tangent at S where the two points of intersection lie on corresponding rays from S.

It is assumed in the following direct method used by Reye to find the twenty-seven straight lines on a cubic surface, that if two twisted cubics have a common point S, and are generated by projective pencils S, S′, and S, S″ the twisted cubics have two and only two common secants.

It may be easily shewn that they have not more than two unless they have an infinite number, for if they have three common secants and the point S common, they must lie on the same hyperboloid determined by the three secants, in which case an infinite number of the generators of the hyperboloid will meet the two twisted cubics. In this case the cubic surface would degenerate into a plane, and a surface of the second degree, which is contrary to supposition.

Let a point Q move along the second twisted cubic, and let secants be drawn from different positions of Q to the first twisted cubic, we know that each position gives a definite secant, and therefore a surface will be generated that gives a plane curve on a plane section.

Next let a point q move along the first twisted cubic and determine a curve on the same plane section by secants drawn to the second twisted cubic. These curves will have a certain number of points of intersection, and it is clear that to each point of intersection corresponds a common secant. If we do not include the secants corresponding to the point S, we know there cannot be more than two, and that there are two we can shew by special cases. For example, let the twisted cubics each degenerate into three straight lines, two of these will intersect at S, and the remaining pairs will reduce to four non-intersecting straight lines which have two and only two common secants.

In both cases, that is, whether the curves are generated by points moving on twisted cubics, or whether these twisted cubics have degenerated into straight lines, their degree is the same, and we obtain the same number of intersections.

Hence we may infer that there are two common secants and not more than two.

Reye finds the twenty-seven straight lines on a cubic surface by correspondence.

Let a cubic surface be generated by three projective pencils at centres S, S′, S″, and let any plane through S correspond to a straight line in a plane P. We see that this is possible, for any plane in the pencil S may be fixed by its intersection with P in a straight line. Now every plane of the pencil S corresponds to planes in the other pencils which by their intersection fix a point on the surface. Therefore we infer that unless the three corresponding planes intersect in a straight line every point on the surface corresponds uniquely to a straight line in P.

Next take another plane P′ so that every point in P′ corresponds to a straight line in P. This is reciprocal correspondence, for as two points in P′ lie on a straight line, so the corresponding straight lines in P intersect in a point. We may take as an example the properties of pole and polar.

We finally arrive at the conclusion that to every point on the surface corresponds one, and only one point on P′, provided the three corresponding planes of the pencils meet at a point.

I. Every straight line on P′ corresponds to a twisted cubic on the surface, for every straight line on P′ determines a number of straight lines through a point on P, which in turn determine three axial pencils, which by their intersections fix a twisted cubic on the surface. This is not the same kind as the twisted cubic on p. 35, and is said to be of the second species.

II. Every plane section of the surface corresponds to a plane cubic curve on P′, for every straight line in P′ must meet a curve which corresponds in degree with the order of the curve which it represents on the surface, and we know that a twisted cubic is of the third order.

III. Every straight line on the surface must be represented by part of a cubic curve on P′, for it represents part of a plane section of the surface.

Next we infer that if we take two twisted cubics on the surfaces generated by pencils having centres at S, S′, and S′, S″, then these twisted cubics have two common secants, and that as these imply that corresponding planes meet in the same straight line, and not at a point, we find that in this case a point in P′ corresponds not to a point, but to a straight line on the surface.

Taking the twisted cubics two and two we have three pairs of points, that is six points in P′ that correspond to six non-intersecting straight lines on the surface. The six points in P′ do not lie on the same conic, they are called principal points. Now let a straight line in P′ pass through Q, a principal point. This straight line represents a twisted cubic on the surface of which one point represents a straight line on the surface, therefore the remaining points on the straight line through Q correspond to a conic on the surface.

Now this conic on the surface is part of a plane section, the remainder being a straight line LM. Since every plane section on the surface must cut the straight lines represented by the principal points, therefore every cubic on P′ representing a plane section of the surface must pass through the six principal points. Next LM and the conic make up a plane section of the surface where the conic is represented by a straight line in P′ passing through Q. Therefore LM must be represented in P′ by a conic passing through the five other principal points. Therefore LM must meet five straight lines of the six non-intersecting straight lines on the surface represented by the principal points in P′. But this is true for any five of the six straight lines. Hence we get a double six on the surface. The remainder of the straight lines on the surface and their properties are found as before. Or we may proceed thus: join two principal points Q and R in P′, then the straight line QR corresponds to a twisted cubic on the surface, *of which two points* Q *and* R *represent two non-intersecting straight lines*, therefore the other points in QR correspond to a straight line on the surface. In other words the twisted cubic on the surface corresponding to QR degenerates into three straight lines.

Taking the fifteen pairs of the principal points we find the fifteen other straight lines. We also thus obtain the intersections of the fifteen straight lines among themselves, for every straight line as QR meets those of the other fifteen, the corresponding lines of which in P' do not pass through Q or R.

From the definition given of a cubic surface we derive the following construction (Reye).

If four faces of a variable tetrahedron turn about four fixed points, and three of its vertices move along three straight lines passing through a point, then the fourth vertex traces out a cubic surface. For we note that the plane faces of the tetrahedron generate pencils, three of which being perspective to the fourth are projective to one another, and therefore generate a cubic surface by the intersection of their corresponding planes.

Dr Salmon gives a very similar construction for twisted cubics, the four faces of the tetrahedron turn about fixed lines instead of passing through fixed points. It is evident that if the "fixed lines" in the second of these constructions pass through the "fixed points" of the first, the twisted cubic lies upon the cubic surface.

From the first of these we derive a simple construction given by Grassmann for a plane cubic curve.

Let the three fixed points S, S_1, S_2, be the points about which the three faces of the tetrahedron move, which by their intersection generate the surface.

Let the three fixed lines meet the plane SS_1S_2 in A, B, C, and let them intersect in O.

If we project the edges of the tetrahedron from O upon the plane SS_1S_2, three of the projected edges become the sides of the triangle ABC, and are fixed in position. If P be a point on the surface in the plane SS_1S_2, then PS, PS_1, PS_2 will meet the sides of ABC in three points, all of which lie in the fourth face of the tetrahedron, and must be in one straight line.

Hence we infer that if a point be taken so that the straight lines joining it to the angular points of one triangle meet it on the sides of a second triangle, and the three points of intersection lie in one straight line, then the point moves in a plane section of a cubic surface.

By proper selection of S, S_1, and S_2, because we can generate the surface by three points lying in any plane section, the construction applies to any plane section of the surface.

If we take Grassmann's construction, and use the theorem that the ratio compounded of the ratios of alternate segments of the sides of a triangle by a straight line is unity, we find that the anharmonic ratios $P\,[SBS_1C]$ and $P\,[S_2AS_1C]$ are equal.

This result is given by Dr Salmon, but appears in another form in Reye's *Geometry of Position.*

He shews that a plane cubic can be generated by a pencil of rays from a point on the curve as T, and a pencil of conics passing through four points as S, B, S_1, C. To each conic corresponds a ray through T, and these intersect on the cubic.

Now take the methods of tracing by the two conics. Describe any conic through S, B, S_1, C, and take another having the same anharmonic ratio through S_2, A, S_1, C, that is to say that if X be any point on the first and Y any point on the second, then

$$X\,[SBS_1C] = Y\,[S_2AS_1C].$$

The relation is unique: to every conic of one system corresponds one, and only one of the other. Let these conics intersect in P and P_1, then both these points are on the cubic and the chord PP_1 passes through a fixed point T. So that we might have drawn the chord TPP_1 and supposed P and P_1 found by its intersection with one of either of the series of conics, through SBS_1C or through S_2AS_1C.

To proceed, however, with the statement that in a plane cubic we can take two sets of points SBS_1C and S_2AS_1C, and suppose the cubic generated by the point P where $P\,[SBS_1C] = P\,[S_2AS_1C]$.

Project the curve so that S_1 and C are the circular points at infinity, then we find that two similar segments of circles described upon fixed straight lines generate a cubic curve.

If we take these fixed straight lines as parallel and bisected at right angles by the same straight line we obtain a symmetrical circular cubic.

Many interesting properties can be proved about this curve, which in turn can be projected into the case of the general cubic.

From the construction $P\,[SBS_1C] = P\,[S_2AS_1C]$ we may infer that if the curve has four points in one straight line, it is made up of this straight line and a conic. For if S, B, S_1, C are in one straight line, then P moves either in this line, or in a conic $P\,[S_2AS_1C]$, for the anharmonic ratio to four points is now constant. Similarly if the ratio $P\,[S_2AS_1C]$ were constant, then S, B, S_1, C not being on the conic must lie on one straight line. But this ratio would be constant if more than six points were common to the surface and a conic, for if we have a seventh point common we find by taking two cases that S and B must also lie on the same conic or lie in one straight line.

We may say then that a plane cubic will degenerate into a straight line and conic if more than three points are in one straight line, or more than six on a conic.

Reye uses the proposition that a plane cubic curve may be generated by a pencil of lines and a pencil of conics to prove that an infinite number of cubics passing through eight points will also pass through a ninth fixed point. (French translation, p. 226.)

"Among the given points we take four, of which no three lie in one straight line, we cause to pass through them a pencil of conics, of which we select five, *a*, *b*, *c*, *d*, *e*. The four points being O, P, Q, R, we call the pencil of conics [OPQR]. The five conics *a*, *b*, *c*, *d*, *e* may be made to pass through five arbitrary points A, B, C, D, E.

"We can select rays DA, DB, DC which correspond respectively to the conics in the pencil of conics that pass through A, B, C. [Compare with the analytical results p. 30 with p. 41.] We can then draw rays DD_1 and DE_1 of the pencil D which correspond to the conics *d* and *e*. We may suppose a conic drawn through

A, B, C, D, touching at D the ray DD_1, denote it by k. It is cut by the ray DE_1 in a point E_1, and is projective through the pencil D to the pencil of conics [OPQR], corresponding in the same way that a, b, c, d, e do to the points A, B, C, D, E_1 of the conic k.

"Every pencil of rays S, perspective to the conic k, generates with the pencil of conics [OPQR] a curve of the third order which passes through the eight points O, P, Q, R, A, B, C, D. If we take upon the conic k the centre of the pencil of rays S so that the point E_1 is projected by the ray SE, the curve of the third order passes also through the ninth point E. For one position of the point S the curve of the third order has in common with the conic the five points A, B, C, D, S, and consequently has yet a sixth point T. This point T is situated in the same way upon a conic of the pencil [OPQR] as A, B, C, and D are, consequently it is contained by each of the system of curves of the third degree that passes through A, B, C, D, O, P, Q, R.

"It is only when E coincides with T that all the curves pass through E."

Reye remarks that the points E_1 and S may be found by linear construction by use of Pascal's theorem.

On Tangency.

In a cubic surface generated by three projective pencils at S, S_1, and S_2 take at S the plane that corresponds to the point S. It is clear that we may do this, for every point on the surface is the intersection of three corresponding planes, one from each centre. When we draw from S_1 or S_2 the corresponding planes, they cut it in secants of the twisted cubics SS_1 and SS_2 that meet the twisted cubics in two coincident points at S—see the article on secants of twisted cubics.

Since S_1 and S_2 are arbitrary, every twisted cubic on the cubic surface passing through S has a secant cutting it at S in two coincident points, and these secants lie in one plane.

It is clear then that any plane section cutting the plane at S in one of these secants has this secant for its tangent at S. For the secant cuts the surface at two coincident points at S, and meets it again in one point only.

This may also be seen in another way. Let any plane section cut the surface, passing through the point S, and let us take the sections also of the cones generating a twisted cubic passing through S. The secant in the section lies in the tangent planes to the cones at S, but the sections of these cones are conics that intersect on the surface, and since they have the secant as tangent at S, this secant must also be a tangent to the cubic curve.

We see then that the tangents to all twisted cubics on the surface passing through S, and also those to all plane sections at the point S lie in one plane, the plane at S, that corresponds to the point S in a pencil that generates the surface.

This plane is called the tangent plane to the surface at S.

If two straight lines on the surface pass through S, since they are tangents at S to sections drawn through S, the tangent plane at S contains these straight lines, and is determined by them.

We see then that a plane through three straight lines on the surface is a tangent plane at the three angular points of this triangle on the surface. It is therefore a triple tangent plane.

CHAPTER III.

MISCELLANEOUS.

To prove by Pascal's Theorem that the straight lines meeting three non-intersecting straight lines generate a conicoid, i.e. a surface, every plane section of which is a conic.

[*Construction.* Let ABCDEF be a plane hexagon, so that the opposite sides AF, CD; FE, BC; ED, AB meet respectively in L, M, N.]

Let P_1, P_2, P_3; p_1, p_2, p_3 be two sets of non-intersecting straight lines in space, each line of one set meeting all of the other set, let them meet any plane in the points A, C, E, B, D, F. The plane is not to contain one of the given straight lines.

Denote the intersections P_1p_2, P_2p_3, P_3p_1 by X, Y, Z. Now X, Y, L lie in one straight line, for they will all be found in both planes P_1p_3 and P_2p_2, and lie on their intersection. Similarly, Y, Z, M lie in the planes P_3p_3 and P_2p_1, and X, Z, N in P_3p_2 and P_1p_1.

Therefore L, M, N are the points in which the sides of the triangle XYZ meet the plane, and are therefore in one straight line, so that A, B, C, D, E, F lie on a conic.

Since five points determine the conic we may consider F and therefore p_3 as variable, and any straight line meeting P_1, P_2, P_3 traces out the conic ABCDE.

Hence we find that two and only two non-intersecting straight lines can be drawn to meet four non-intersecting straight lines P_1, P_2, P_3, P_4. There are two and only two common generators to conicoids drawn through P_1, P_2, P_3, and P_2, P_3, P_4 besides P_2 and P_3.

Any one of the straight lines on a cubic surface is cut by ten others in a range in involution.

Suppose in figure 3 that ee' is the line 13, ED is 15, and AM is 14, and the lines 23, 18, 19, 22, 12, 24, 6 and 25 meet the plane of 13, 14, 15 in e, e', d, d', C, G, B and K respectively.

Since three straight lines in a plane meet the plane in a straight line, and 19, 22, 24; 23, 6, 24; 6, 25, 22; 12, 25, 18 are planes, and AD, AM meet ee' in b and b', we find b, b', e, e', d, d' are the points in which the sides and diagonals of a quadrilateral meet a straight line, therefore they form a range in involution.

Similarly any other of the five pairs besides d, d' form a range in involution with b, b', e, e'.

To place "a double six" in position.

A double six consists of twelve straight lines U, V, W, X, Y, Z, u, v, w, x, y, z; any one of the first group, as U, meets five of the second group v, w, x, y, z, but does not meet any of the remaining lines.

Take a straight line ee' as the intersection of two planes represented by figs. 2 and 3. These planes may make any convenient angle with one another.

The points a, a', b, b', c, c', d, d', e, e', form a range in involution, a being conjugate to a', b to b', and so on.

In 3 take the straight lines Ab and Ab' to represent v and U, and in 2 take A$'c$, A$'c'$ to represent u and V.

Next on Ab and A$'c$ take the points B and B$'$. Let Ba meet Ab' in M, and let B$'a$ meet A$'c'$ in M$'$.

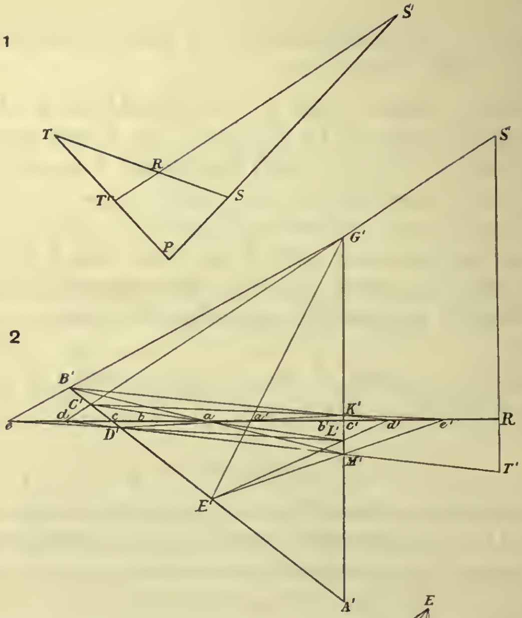
Similarly eB and Bd' give G and K.

e'K produced meets Ab in C, next Ca meets Ab' in L, and Ld' meets Ab in E.

Ka' meets Ab in D.

1

2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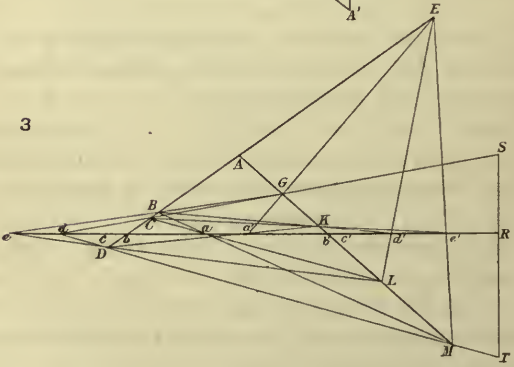
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A similar construction in 2 determines the position of G′, K′, C′, L′, E′, D′.

It remains to be proved that the points C G d, L D e, G E a', D M d, M E e' respectively lie in straight lines.

Considering the quadrilateral CGKB we see that four sides and one diagonal pass through the points b, b', e, e', d', therefore the remaining diagonal CG must pass through d which is conjugate to d' in a range in involution, for it is known that the sides and diagonals of a quadrilateral cut any straight line in a range of points in involution.

Therefore the points C, G, d are in one straight line. Taking the quadrilateral LDCK we can prove L, D, e to be in one straight line.

GECL may be used to prove for G E a', DMBK for D M d, and BMGE for M E e'. The same line of proof may be used for the second diagram.

It can now be shewn that the straight lines U, V, W, X, Y, Z, u, v, w, x, y, z may be represented by A b', A′ c', BB′, CC′, DD′, EE′, A′ c, A b, LL′, MM′, GG′, KK′.

No proof is necessary for U, V, u, v, for by construction the points G, K, L, M, &c. lie on these lines, and with regard to the remainder it will be sufficient to prove that one of the straight lines, *e.g.* Z, meets u, v, w, x, y, for the construction will be found symmetrical.

Suppose then we take EE′ or Z.

EE′ meets u and v, for E is on u and E′ on v. Next EE′ meets w or LL′, for L E d', L′E′ d' are two straight lines meeting in d'. Therefore a plane may be drawn to contain both of them. Therefore LL′ and EE′ lie in the same plane and must intersect. In the same manner it may be proved that EE′ meets MM′ and GG′, that is that Z meets x and y.

It is interesting to observe that from this construction all the remaining straight lines on the cubic surface passing through the double six can easily be found.

We take a pair X, x with another Y, y, the intersection of the planes X y, x Y is a straight line on the surface.

One point on this straight line is clearly d. To find another take a plane (fig. 1) at right angles to ee' meeting 3 in SRT and 2 in S′RT′, then SRS′ in 1 represents the angle of inclination of 3 and 2.

The lengths RS, RS′, RT, RT′ are measured from 3 and 2, and set off in 1.

Join SS′, TT′, they meet in P the second point required.

We thus find six straight lines on the surface passing through a, d, e, a', d', e'. ee' itself is also a straight line on the surface.

The only straight lines now remaining are those of which the intersection of the planes Uz, uZ is an example, that is Ab', KK′ with A′c, EE′.

One point of this straight line in diagram 3 must be at the intersection of Ec with Ab', and another in diagram 2 at the intersection of A′c with b'K′.

When a double six has been placed in position all the remainder of the 135 points of intersection of the twenty-seven straight lines on a cubic surface can be found by methods indicated above, but if we allow that a cubic surface passes through every point of a double six, the remaining straight lines with their points of intersection can be found by use of the propositions stated on page 29, as in Chapter I. p. 26. In other words, we first state that a straight line that meets four straight lines on the surface must itself lie on the surface, and secondly that if six lines a, b, c, a', b', c' are so related that abc, $a'b'c'$ form plane triangles, where a meets a', b meets b', then c meets c', the three points of intersection lying in one straight line.

If we take a section of this system of lines allowing only that they are placed according to the table of reference p. 3, we can shew that if 10 points of intersection lie on a straight line, which is obviously a line of the system, then the remaining 16 points lie on a conic, for we can take successive groups of six points, and by use of Pascal's Theorem shew that they lie on a conic, because sets of three points corresponding to three lines in a plane are so related that one is on the given straight line, and the other two in a line with it, so that opposite sides of successive hexagons intersect by pairs on the same straight line.

On the coincidence of straight lines on the surface.

If we take any pair of straight lines on the surface it is possible to select a double six to which this pair belongs. If we suppose two intersecting straight lines to coincide while the others remain different, but intersect as before, it is obvious that straight lines taken as non-intersectors now intersect, which is contrary to supposition. Since however we know that two straight lines drawn to meet four non-intersectors are either different, coincident or imaginary, it is not contrary to supposition to take two non-intersectors as coincident.

Even this supposition, when certain lines are given, causes the geometrical construction to become indeterminate. Therefore from a geometrical point of view it is safer to suppose all the twenty-seven straight lines to be real and different, than to suppose certain lines to move nearer to one another without breaking continuity, in other words still intersecting according to the table of reference. It must also be remembered that not more than three straight lines must lie in one plane, and not more than two straight lines must meet four non-intersecting lines. We suppose points of intersection to become indefinitely near without becoming actually coincident.

The simplest case of coincidence is that of two straight lines 16 and 17, while no coincidence takes place in lines numbered 1 to 15.

When 16 coincides with 17 we see that 18 must coincide with 19, otherwise we get four straight lines in a plane, namely 16, 18, 19, and 1, and 1 is not supposed to coincide with either of the others.

By similar reasoning we find that 20, 21; 22, 23; 24, 25; 26, 27 coincide by pairs.

We will proceed to prove that these rays of double lines lie on a cone of the second degree. It is clear that they all pass through one point. It may be proved by experiment that any supposition of coincidence reduces the general table of reference to one of the others. This subject will be considered in Chapter VII. from another point of view.

Denote the rays 16, 17; 18, 19; 20, 21; 22, 23; 24, 25; 26, 27 by the letters r_1, r_2, r_3, r_4, r_5, r_6.

Construct a triangle representing the straight lines 1, 2, 3 and mark points at which $r_1, r_2 \dots r_6$ meet them as $[r_1], [r_2] \dots [r_6]$. We note that $r_1 r_2$ meet 1, $r_3 r_4$ meet 2 and $r_5 r_6$ meet 3.

It will be found convenient in constructing a figure to place the points $[r_1], [r_2] \dots [r_6]$ on the circumference of any convenient circle cutting the lines 1, 2, 3 in the six given points.

Since the lines 6, r_4, r_5 form a triangle we mark the point [6] on the line 1 where the straight line $[r_4]\,[r_5]$ meets it. Similarly $r_2 r_3 5$ gives [5] on 3, and $r_6 r_1 4$ gives [4] on 2.

Now we know that these straight lines 4, 5, 6 make up a triangular section, therefore the points [4], [5], [6] lie in one straight line.

We find then that the opposite sides of the hexagon which is a plane section of the pyramid formed by the six rays are so related that they intersect by pairs that lie in a straight line, and therefore by Pascal's Theorem lie on a cone of the second degree.

This property of the rays of a conic node enables us to determine the form of the surface about the node, for it shews that the shape must be that of a cone of the second degree.

From these relations we can deduce a construction for the lines on the surface.

Take a section of the surface made by the lines 1, 2, 3, mark on these lines $[r_1]$, $[r_2]$, ... $[r_6]$ the points at which the rays meet them. By using the table of reference, noting that where three lines are in a plane their corresponding points on 1, 2, 3 lie in a straight line, we find all the other points of intersection of the lines 4 to 15 with 1, 2, or 3.

The points in which the rays meet the plane must lie on a conic. It will be seen that this condition ensures that three such points as (4), (5), (6) should lie in a straight line by Pascal's Theorem, so that the fifteen lines comply with the conditions of the table of reference.

Next take another section that meets the line 1 as 1, 9, 11 (9 and 11 are arbitrary), and one point at which one of the rays r_3, r_4, r_5, r_6 meets the section. The points on 1 must be marked from the section 1, 2, 3. Using the table of reference as before we can find the points at which the other rays meet the section as well as second points on all straight lines that do not meet 1.

Where the straight lines meet 1 we obtain second points upon them either by taking a third section of lines already found, or as in the general case figure 1, p. 52.

These data may be used to determine the equation to the surface thus—we know that the form of the equation is $uvw = \mathsf{K}x\mathsf{S}$ where $\mathsf{S} = 0$ represents a cone of the second degree.

The six rays lie on the cone $\mathsf{S} = 0$ and thus give its equation. The rays lie by pairs in the planes $u = 0$, $v = 0$, $w = 0$, the plane $x = 0$ must be given as the plane of 1 2 3. The pairs of rays being so selected that r_1, r_2 lie in $u = 0$, r_3, r_4 in $v = 0$ and r_5, r_6 in $w = 0$. The constant alone remains and that is found by any one arbitrary point on the surface, or by any straight line on the surface which requires only one additional constant to fix its position.

Geometrical construction for a surface having a binode B_3.

Comparing the tables of reference in the last chapter for the conic node and binode, we see that the only difference in the construction for the binode and the conic node is that instead of taking r_1, r_2, ... r_6 as lying on a cone, they lie in two planes, so the points $[r_1]$, $[r_2]$ ... $[r_6]$ lie by threes in two straight lines arbitrarily taken in the plane of 1 2 3 but not intersecting on a side.

The construction to find the remaining lines is much the same as in the case of the conic node.

Geometrical construction for lines on a surface having a binode B_4.

If we refer to the table of reference it is obvious that we should first take the triangle *tbc*, and on *t* mark a point (E), and on *b* and *c* mark (D) and (B). The straight lines (B) (E) (C) and (D) (E) (A) give (C) and (A) on *b* and *c*. Similarly (B) (D) (*a*) and (C) (A) (*d*) give (*a*) and (*d*) upon *t*.

Next take the arbitrary triangle *tad*, but mark upon *t* all the points (*a*), (*b*), (*c*), (*d*) and (E) found in the first section. We need only take the point (A) arbitrarily in this section, for (*c*) (A) (B) gives (B), (A) (E) (D) gives (D), and (E) (B) (C) gives (C).

We have thus found two points on each of the rays.

The rays intersect in the node through which E also passes. The position of all the rays and the edge is fixed.

Geometrical construction for lines on a surface having two conic nodes.

Let the points $[r_1]$, $[r_2]$ in the case of one conic node move up to one another and coincide.

Two points on each straight line can be obtained exactly by the same method as that used in the case of one conic node.

Geometrical construction for lines on a surface having three conic nodes.

Take a plane triangle to represent $t_1 t_2 t_3$. [See table of reference for three conic nodes in Chap. I. p. 16.]

On these lines mark points $[A_1]$, $[A_2]$, $[A_3]$ at which the axes meet them. Take a point $[R_1]$ at which a ray meets t_1. The points of intersection of the remaining rays are found on t_1, t_2 and t_3 from the planes $R_1 A_3 R_3$, $R_2 A_1 R_3$, $R_1 A_3 r_3$, $r_3 A_1 r_2$, $R_3 A_2 r_1$.

It will be observed that the points $[A_1]$, $[A_2]$, $[A_3]$ must in the first instance be taken in one straight line, because the axes are in one plane.

Subject to this condition the axes may be taken in any arbitrary position. The intersections of the axes give the nodes which give second points on each ray.

The case of four conic nodes has already been discussed in Chapter I. The constructions for two binodes, and three binodes B_3 are not difficult, but the form of the equations to the surface can not be discovered by the position of the lines on the surface alone.

The construction for B_3 + one conic node C_2 is very similar to that for two conic nodes.

In the remaining cases we again find that the lines on the surface can easily be placed in position, but the form of the surface must be found from other properties of the nodes.

On the number of parabolas upon the surface, and plane sections through a straight line.

It has been stated (p. 30) that if we take the equation to the cubic surface as $uvw = kxyz$, and if we take a plane section through $x=0$, $u=0$ the conic which completes the section is found from the equations $u = mx$ and $mvw = kyz$.

If we express the condition that the conic represents two straight lines, we find m involved in the fifth degree, but if we make the conic a parabola, m is involved in the fourth degree. Thus we have four parabolas of which two or all may be imaginary.

In some cases the parabola coincides with a section known already to be two straight lines, but as this only means that terms of the second degree must be a complete square, we infer that the straight lines are coincident or parallel. Again two parabolic sections may coincide, and we appear to have three parabolas or one.

In this case it is well to note the order in which the sections take place. Usually we have hyperbola, parabola, ellipse, but if two parabolas coincide we probably have hyperbola, parabola, hyperbola.

Also when passing through the phase two straight lines the changes are as follows—hyperbola, two straight lines, a conjugate hyperbola,—that is to say if we take a section represented by two straight lines and turn the plane through a very small angle about $x=0$, $u=0$ we get a hyperbola the asymptotes of which very nearly coincide with the straight lines. If the plane were turned through a small angle in the opposite direction, the section would again be a hyperbola, having the same asymptotes but conjugate to the first. The curvature of the section changing everywhere from convex to concave in regarding the two hyperbolic sections.

We note an important exception. When two sections represented by straight lines move up to and coincide with one another, then the hyperbola caused by moving the section through a small angle in either direction is of the same kind, and not conjugate. The reason is clear when the triangular sections have coincided, the intersection of the pairs of coincident lines is a conic node, and therefore the sections approximate to those of a cone of the second degree. We arrive at the same result by counting the changes thus—hyperbola, straight lines, conjugate, straight lines, hyperbola; when the straight lines coincide, the conjugate disappears.

This subject will be considered later with reference to plane sections of models.

See Chapter VII. and figure 5, Chapter VI.

In these sections B and C represent a hyperbola, and its conjugate, while A represents two straight lines, which in each case make a complete section with the horizontal straight line. Sections of the form B or C are obtained in the general case, by turning the plane round the horizontal line in A through a small angle in opposite directions. When there is a node, C_2, the same kind of section B or C is obtained in either direction.

On the number of points necessary to determine a cubic surface.

We know that the equation of a cubic surface and therefore the twenty-seven straight lines upon it may be determined by 19 points upon the surface, not more than 9 being in the same plane, but a less number of points will be required if they are related to the surface in any special manner.

To take an illustration from conic sections.

Usually five points are required to determine a conic, but if we state that two given points are the extremities of the major axis we actually give four points, so only one other arbitrary point is required.

The tetrahedron in the following construction is taken from a figure in a paper by Mr H. M. Taylor already referred to.

Take the tetrahedron as ABCD cut by four planes

$$\begin{aligned} -lx + \mathrm{M}y + \mathrm{N}z + \mathrm{P}u &= 0, \\ \mathrm{L}x - my + \mathrm{N}z + \mathrm{P}u &= 0, \\ \mathrm{L}x + \mathrm{M}y - nz + \mathrm{P}u &= 0, \\ \mathrm{L}x + \mathrm{M}y + \mathrm{N}z - pu &= 0, \end{aligned}$$

the planes of the tetrahedron being $x=0$, $y=0$, $z=0$, $u=0$.

The lines 1, 2, 3 are the intersections of $y=0$, $z=0$, $u=0$ with $-lx + \mathrm{M}y + \mathrm{N}z + \mathrm{P}u = 0$;

4, 5, 6 of $x=0$, $z=0$, $u=0$ with $\mathrm{L}x - my + \mathrm{N}z + \mathrm{P}u = 0$;

7, 8, 9 of $x=0$, $y=0$, $u=0$ with $\mathrm{L}x + \mathrm{M}y - nz + \mathrm{P}u = 0$;

10, 11, 12 of $x=0$, $y=0$, $z=0$ with $\mathrm{L}x + \mathrm{M}y + \mathrm{N}z - pu = 0$.

The general equation may be put into the form

$$(\mathrm{L}+l)(\mathrm{M}+m)(\mathrm{N}+n)(\mathrm{P}+p)\,xyzu = (-lx + \mathrm{M}y + \mathrm{N}z + \mathrm{P}u) \times \\ (\mathrm{L}x - my + \mathrm{N}z + \mathrm{P}u)(\mathrm{L}x + \mathrm{M}y - nz + \mathrm{P}u)(\mathrm{L}x + \mathrm{M}y + \mathrm{N}z - pu).$$

It is evident that this surface can be constructed if we determine the ratios the constants L, M, N, P, l, m, n, p bear to one another.

If the tetrahedron of reference be a regular one, having all the edges equal, certain points of intersection seven in number taken upon the edges of the tetrahedron divide them in the ratios of these constants. If the tetrahedron be not regular these points still determine the ratios of the constants, but the lengths of the edges have to be considered.

Take the tetrahedron as regular, the base being ADC, B the centre of the triangle marking the projection of the vertex of the tetrahedron upon the plane of ADC.

In BD mark the point	(12, 10).	This determines the ratio M : p.
In BD produced	(2, 8).	................................M : P.
In AD produced	(4, 9).	................................L : P.
In DC	(8, 9).	................................n : P.
In DC produced	(6, 1).	................................N : P.
In AC	(1, 3).	................................l : N.
In BA	(4, 5).	................................L : m.

(4, 9), (4, 5)	produced gives	(4, 6) in BD,
(6, 1), (1, 3)	....	(1, 2) in AD,
(8, 2), (2, 1)	....	(2, 3) in AB,
(4, 9), (8, 9)	....	(7, 9) in AC,
(8, 2), (8, 9)	....	(7, 8) in BC,
(7, 9), (7, 8)	....	(7, 12) in AB,
(7, 12), (10, 12)	....	(11, 12) in AD,
(6, 1), (4, 6)	....	(5, 6) in BC,
(5, 6), (4, 5)	....	(5, 11) in AC,
(5, 11), (11, 12)	....	(10, 11) in DC,
(1, 3), (2, 3)	....	(3, 10) in BC.

13, 14, 15 are found as the intersections of known planes, *e.g.* the planes (4, 13, 9) and (13, 10, 3) are known because 4, 9, 10, 3 have been found, and the intersection of these planes is 13.

The numbers 16 to 27 cannot in this case be found by geometrical construction, but if one point on one of them is determined by trial, or by a quadratic derived from the equation, the remainder can be placed in position.

We note that though the seven points are connected with an analytical equation, the finding of the points of intersection of fifteen straight lines from seven given points is entirely geometrical.

CHAPTER IV.

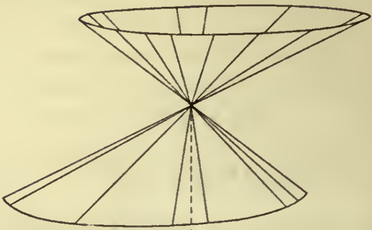
ON MODELS OF CUBIC CONES.

WE may suppose a model of a cubic surface to be built up by a number of parallel horizontal sections each of which is a plane cubic curve.

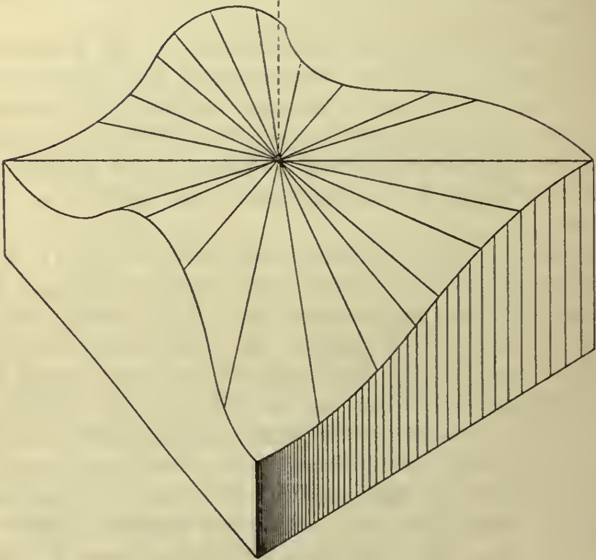
Now generally if the intersections of the twenty-seven straight lines on the surface take place at finite distances from one another, and if parallel sections be drawn at infinite distances above and below the model, these sections are equal and similar but reversed in position. Further if the surface be cut out of a solid block the hollow and solid portions are interchanged.

Figures 3 and 4 are horizontal sections of the model of a cubic surface (Chapter VI.) at a very great distance above and below it. Now we observe the following facts: the curves are equal and similar but reversed in position, the oval is solid in the first (figure 3) upper section and hollow in the lower (figure 4), so also are the hyperbolic portions solid in figure 3 and hollow in figure 4. In other words the surface approximates to a cubic cone. A sketch of the sheets of such a cone is given on page 64. The dotted line indicates that the vertices of both sheets are at the same point, they are drawn separate because it is difficult to shew the lower part of the first sheet within the second.

When carved out of a single block the upper part of the first sheet is a solid cone, each horizontal section of which is an oval

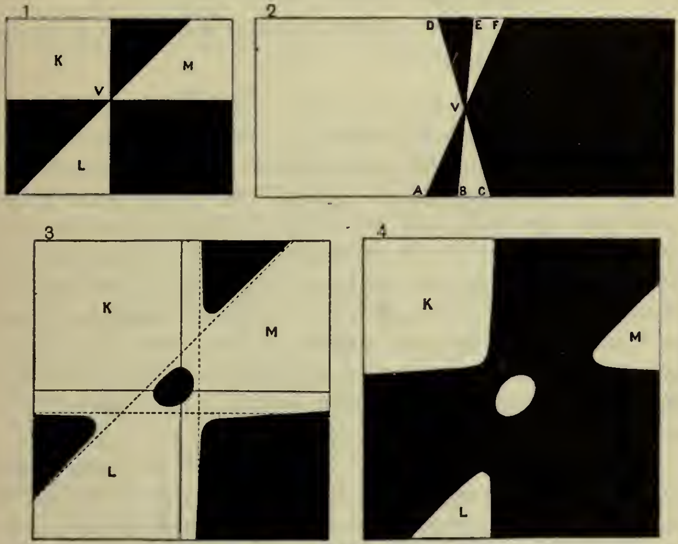


First sheet of surface of cone.



Second sheet of surface of cone.

(figure 3); the lower part is a hollow cone, the sections of which give the hollow oval (figure 4), while the sections of the second sheet in both cases give the hyperbolic branches. Looking at the second sheet we see why this is so; it consists of three elevations and three depressions; a line of greatest elevation of a raised portion being produced through the vertex gives the line of greatest depression in the opposite direction. Figure 2 shews a vertical section taken diagonally across the horizontal sections



shewn by figures 3 and 4, VDE being a vertical section of the upper conical portion and VBC of the lower hollow cone. The vertical section is taken in each case across figures 3 and 4 from the corner of the portion marked K at right angles to the oblique asymptote. AB is a section of the second sheet between K and the oval. VF marks approximately a line of greatest elevation, and VA of greatest depression. Both sheets of the cone pass through V. The horizontal section through V is shewn on figure 1. The sections immediately above and below change in curvature from convex to concave, and *vice versâ*, with reference to solid and hollow portions. Thus take parts of sections marked K, L, M. Above figure 1, as in figure 3, the solid portions bounding K are convex hyperbolic branches, and the oval is convex. Below figure 1, as in figure 4, K is bounded by a concave solid portion which we may also call a convex hollow curve. In figure 1 the boundaries are neither convex nor concave, being straight lines.

To describe the model of such a cone as that shewn in figures 1, 2, 3, take its equation as

$$xy(x+y-nz)=z^2(\mathsf{L}x+\mathsf{M}y+\mathsf{N}z).$$

To ensure an oval the values of L, M, N must be so taken that if $z=1$ a point on the curve must lie within the asymptotic triangle $x=0$, $y=0$, $x+y=n$. Take any convenient value for z and trace sections, such as figures 3 and 4, upon two thin pieces of wood. Cut out the curves in the lower section (figure 4), but only pierce holes at convenient distances on the curve in figure 3. Fix the pieces of wood at any convenient distance apart horizontally by side supports. Pass wires or threads through the holes in the upper section and tie them down to corresponding points in the lower section. We thus obtain a cone of threads.

A model of plastocene may be made on these as generating lines; then we find as required a solid upper cone, and a hollow lower one, with a second sheet from which a horizontal plane would cut three solid hyperbolic branches above the vertex and three hollow ones below it. If there is no oval the cone consists of the second sheet only.

It is convenient to take two asymptotes at right angles, because for any given value of x we only have to solve a quadratic, not a cubic for y.

The special cases are: (1) the oval may contract to a point, (2) the curve may have a node, or (3) it may have a cusp.

CHAPTER V.

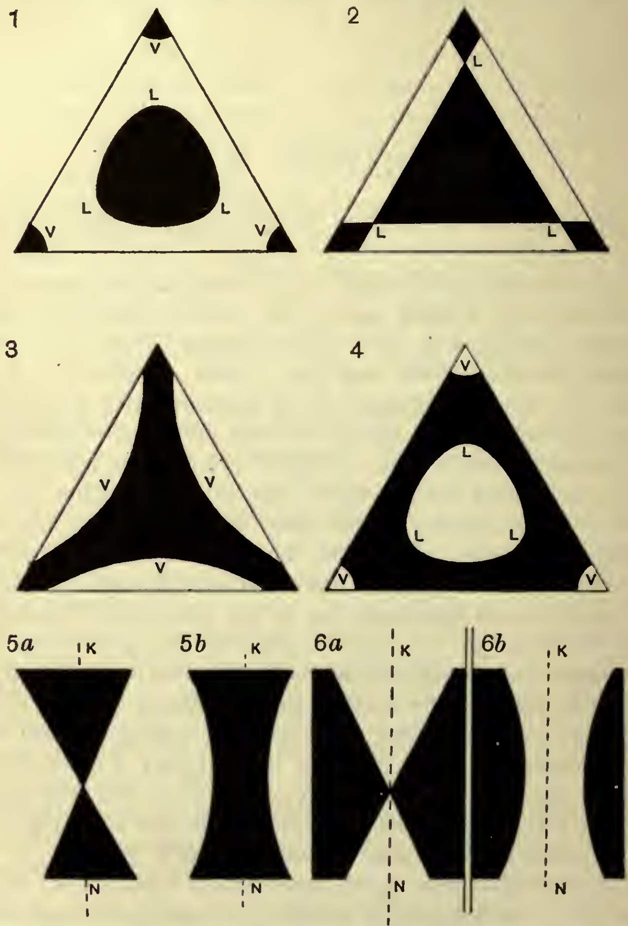
ON MODELS OF SURFACES WITH FOUR CONIC NODES.

As in the case of the cubic cone surfaces may be represented by building up a solid model. Fix certain wires, or solid portions in wood, and cover with plastocene, or some other modelling material. We thus get a model as definite in form as a sphere or as an ellipsoid. It is necessary then to consider space divided by a cubic surface into two parts, hollow and solid.

Usually a straight line cutting a surface, thus represented, passing at infinity through solid space in one direction lies in hollow space at infinity in the other direction, for it cuts the surface three times. When one of the three points of intersection lies at infinity the straight line only appears to cut the surface twice, and apparently lies in the same kind of space in both directions.

As before explained two sections may be the same in form yet differently represented. Take for instance figures 1 and 4, p. 68; these represent plane sections of a cubic surface, an oval and three infinite hyperbolic branches, a small part of each of the branches being shewn in the figures. Now in the first figure the oval and the branches are solid, but in the second hollow, yet the same equation may be used for both.

It will be convenient to call these sections "unlike" but of the same kind. Sections will be called of the same kind when their asymptotes are parallel, and when they are of the same form, *i.e.* as above an oval and three branches, or when two sections consist of a straight line and an ellipse, or of a straight line and a hyperbola. Parallel sections have parallel asymptotes. Take



a section of a cubic surface consisting of three straight lines, none of which passes through a conic node, and are therefore mere lines. Straight lines on the surface that may be considered as formed by the coincidence of two or more straight lines pass through nodes, and are called nodal rays.

If we take this section (figure 2) and move the plane of section parallel to it above and then below through small distances, we obtain sections of the form (figure 1) and (figure 3), or *vice versâ*. The curvature of these sections is opposite. In figure 1 the solid portions are convex, in the latter concave.

This rule does not apply when the triangle is formed by three nodal rays, or more strictly it probably does not apply; we have to consider how many triangular sections have coincided to form the section in question.

In the case of a surface having four conic nodes, each of these nodes (called by Cayley C_2) approximates to a cone of the second degree. There are other nodes of a higher order to be considered later. The shape of these nodes (C_2) is shewn in figures $5a$ and $6a$. Every section through a certain straight line as KN is of the form $5a$ and $6a$ respectively when very near the node.

We will call $5a$ a solid node and $6a$ a hollow node.

When the equation to the surface is slightly altered by varying the constants, or adding other terms with small coefficients, the node does not "break." The solid node takes the form $5b$ and the hollow node that of $6b$, every section about KN being approximately the same.

In the first case we have a solid, and in the second a hollow tunnel.

It is evident from the shape of the nodes that a plane section parallel to the triangle formed by joining three conic nodes (C_2) and near to it has the same curvature whether on one side of it or the other, which is different from the case of a triangle formed by three mere lines.

In drawing sections for models it is often found convenient to take a regular tetrahedron as tetrahedron of reference, the sides of the base being straight lines on the surface which is if possible taken as symmetrical with respect to the sides of the base. The equation to any section parallel to the base may then be written

$$3(x-p)y^2 = x^3 + 3x^2p - 4p^3 + 4k \text{ .... (I)},$$

using rectangular Cartesian coordinates, the origin being at the centre of the symmetrical figure, and $y=0$ is perpendicular to an asymptote. p and k are constants depending upon the position of the section.

The sections are shewn by figures 1 to 4; there is also the case of three infinite hyperbolic symmetrical branches without an oval.

To determine p and k we take a section perpendicular to the base, and passing through an angle bisecting the opposite side. By measuring ordinates in this section we find the values of x when $y=0$ in equation (I). These values being known p and k can be determined.

For example take figure 16 of this chapter.

This is the vertical section of a cubic surface, every horizontal section of which has an equation of the form of equation (I). Draw a vertical through A.

Any horizontal section cuts this figure in a horizontal line, and distances measured from the vertical through A are the values of x corresponding to the value $y=0$ in equation (I). One of these points is sometimes not within the drawing, but two are enough to fix the values of p and k. We can then trace the horizontal section corresponding to any horizontal line in figure 16.

Thus O is the centre of the oval (figure 1) where the vertical through A in figure 16 meets it. The axis of x in figure 1 cuts the curve in L and V and again at a point T on the oval. OL and OT may be measured as a and $-b$, and OV may be taken as c. Since a, $-b$, and c are the roots of

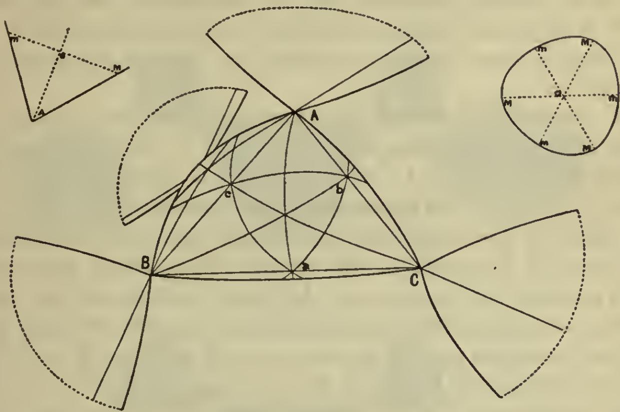
$$x^3 + 3x^2p - 4p^3 + 4k = 0,$$

we find c from $ab - bc + ac = 0$, and p and k from

$$3p = -a + b - c \quad \text{and} \quad 4k - 4p^3 = abc.$$

The curve is an oval and three infinite branches, or three infinite branches simply, except when $k=0$—we then have three straight lines.

It was stated, Chap. I. p. 17, that the shape of a cubic surface in which there are no singularities, and where the straight lines on the surface are real, may be derived from one having four conic nodes, as below.



Again, so long as the nodes do not coincide the properties of the surface are not altered, except by moving the transversal plane into different positions with respect to the nodes.

First then let the transversal plane have all the nodes on the same side of it, and let the plane move to infinity.

The equation to the surface may be written

$$x^2 + y^2 + z^2 = 2xyz + 1,$$

the coordinates of the nodes being (1, 1, 1), (1, -1, -1), (-1, 1, -1), and (-1, -1, 1) in rectangular Cartesian coordinates.

A sketch is given of a model of the surface. It consists of a central solid somewhat like a regular tetrahedron with spherical triangles upon its faces, and four infinite cones one at each node.

Sections of one of these infinite cones are given, the oval being a transverse section perpendicular to a straight line joining the node A to the centre. The other is a plane section through A M m, where M and m are symmetrical points on the oval as shewn in the figure.

The complete section through the oval is made up by three infinite hyperbolic branches cut from the other three infinite cones.

AB, BC, CA are straight lines on the surface ; Aa, Bb, Cc are parts of parabolic sections, while ab, bc, ca are circular arcs.

These sections will be discussed in the construction of a model.

To construct a model take four blocks of wood shaped as equal regular tetrahedra, edge 3 inches, to form the corners of the frame (figure 11), which is itself a regular tetrahedron, edge 14 inches. The figure shews one face of such a figure and the faces of three corners. The fourth corner is not shewn, being behind the model in the centre. By inserting fine screws (with eyes) at the points X in the faces near the middle points of the inner edge of the smaller tetrahedra, and attaching wires to these, arranged parallel to the edges of the frame, we form a regular tetrahedron three of the vertices of which lie at the points A, B, and C. D the fourth vertex of this central tetrahedron lies behind the centre of the triangle ABC. Tie the wires firmly together at A, B, C, D. Make the central solid tetrahedron of plastocene having the wires as edges, and four conical solids at each angle being still guided by the wires. Figure 11 shews also the section passing through the angular points A, B, C. Now commence working up the true shape of the surface on each face. This may be done by drawing sections in cardboard, cutting away the part representing the solid section, and fitting the curve thus obtained to the model. Some of these sections are given, figures 7 to 10. Take a plane containing AB passing also through H the middle point of the edge CD.

The section here (figure 8) is a parabola, together with the straight line AB, and gives a clear view of the nodes at A and B.

Another equal, similar section passes through CD and the middle point of AB, and is at right angles to the plane of ABH.

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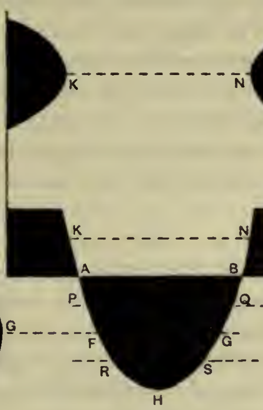

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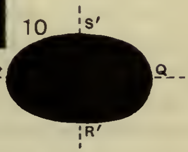

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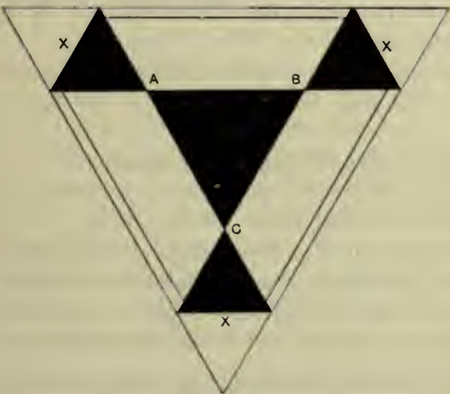

Sections 7, 9, and 10 are at right angles to the planes of both these parabolas, the extremities of the major and minor axes of the sections, which are conics, moving along these parabolas.

Suppose then we take the shape of the surface, section by section, with reference to the parabola AHB (figure 8). Above AB the section at right angles to AHB is a hyperbola, the major axis of which is the double ordinate of the parabola, as KN is the major axis of figure 7. The minor axis of this section is given by the fact that the difference of the squares of the semi major and minor axes is equal to double of the square on the radius of the circle (figure 9). This circle is the central section at right angles to AHB, that is the section through the centre of the tetrahedron ABCD. Its radius is the ordinate of the parabola at the centre.

Next as we come nearer to AB, between KN and AB, the minor axis of the hyperbola diminishes till at AB it vanishes, and the hyperbola becomes two coincident straight lines. On the surface however AB represents four coincident straight lines. Between AB and FG the section becomes an ellipse.

Near AB the minor axis of the ellipse is very small, and is equal to the double ordinate of the parabola in a section equally distant from the centre, at right angles to AHB, and below the centre. For example, $R'S' = RS$ where PQ and RS are equally distant from the centre.

The minor axis of the ellipse continues to increase and the major axis to diminish till at the centre we find the circle, figure 9.

As we might expect the sum of the squares of the semi-major and semi-minor axes of the ellipse in any section is equal to double the square on the radius of the circle.

It is not convenient to take sections at right angles to AHB further, for we can take the same sections over again, working up with the other parabola at right angles to AHB and inverted, passing through the nodes C and D and the middle point of AB.

If we take sections parallel to the triangle ABC (figure 11) we have symmetrical sections (see page 70). In front of the plane of the paper and parallel to it we have three infinite solid branches cut from the conical solids at A, B, C provided the plane does not cut the central tetrahedron. As the plane gets nearer to the centre it must also cut the central tetrahedron, thus giving an oval and three infinite branches. In the plane of the paper $K=0$ and we have the triangular section ABC.

If the section continues to move towards D we still get an oval and three infinite branches, except that at D the oval becomes a point.

The sections obtained as the plane moves further from the centre have already been discussed in describing the sketch of the model.

The equations to all these curves are of the form given on page 70.

The sections at infinity do not become equal, similar and reversed, as stated in the last chapter. The reason is that all points of intersection of lines are not at finite distances, and this was mentioned as a necessary condition. But suppose we take the equation to the surface as

$$D\alpha\beta\gamma + A\delta(\alpha\beta + \beta\gamma + \gamma\alpha) = 0,$$

D and A being of the same sign.

The transversal plane is now parallel to figure 11, but not at infinity, still the nodes are all on the same side of the transversal plane.

The three infinite branches proceeding from the conic nodes A, B, C now unite to form one continuous sheet in the neighbourhood of the transversal plane, and the sections at infinity parallel to the triangle ABC in front of the plane of the paper are a hollow oval and three infinite hollow hyperbolic branches, which are equal, similar and reversed as regards sections at infinity behind the plane of the paper which consist of a solid oval and three infinite solid branches.

All the nodes are of the same kind, being nodes of the same form as double solid cones of the second degree.

In the next surface we can take the transversal plane as parallel to one of the faces of the tetrahedron of reference and bisecting the three remaining edges. At one vertex A, which is

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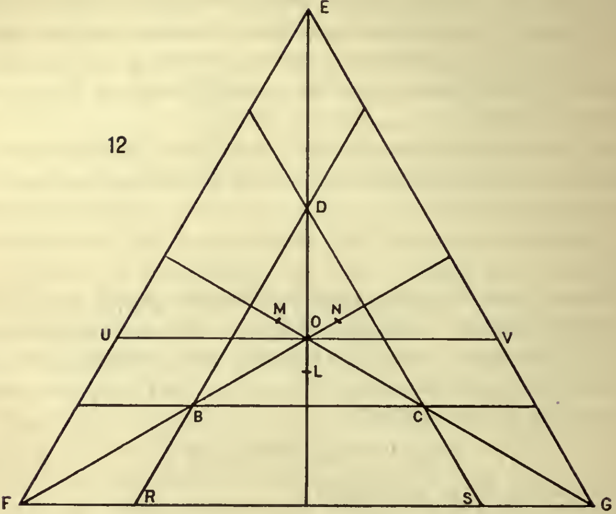

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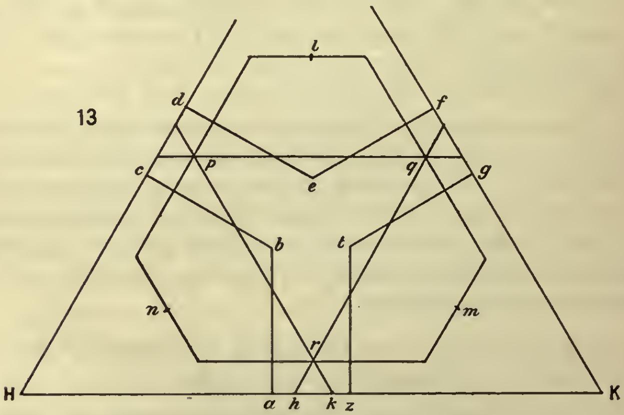

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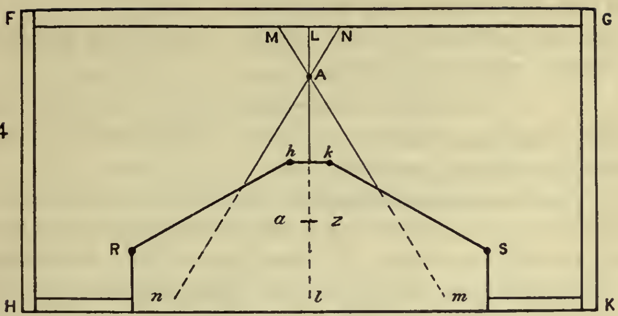

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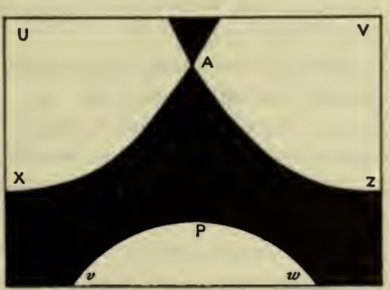

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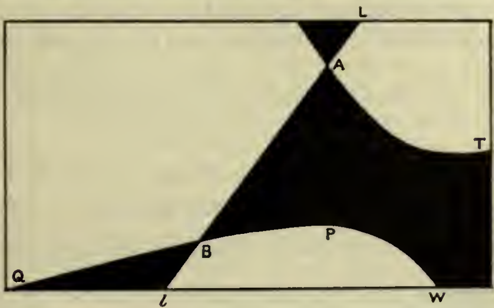

separated from the rest by the transversal plane we have a solid conic node, and hollow ones at B, C, D.

The symmetrical equation to the surface in tetrahedral coordinates may be written

$$\alpha\beta\gamma = \delta(\alpha\beta + \beta\gamma + \gamma\alpha).$$

The equations of the sections may be expressed in Cartesian coordinates without difficulty.

The horizontal sections parallel to $\delta = 0$ are of the form shewn in figures 1 to 4, there being also the case of three infinite branches without an oval. To obtain a clear idea of the sections the infinite branches should be more fully drawn, in the figures 1 to 4 the vertices only are given. Compare figures 3 and 4, Chap. IV. We therefore find vertical sections, and the horizontal sections are then found as on page 70.

Figure 15 represents a vertical section through A bisecting the edges DB and DC of the tetrahedron.

The curve is known to be a nodal cubic, so that if we take rectangular Cartesian coordinates, the origin at A, and the axis of x measured positive in a downward vertical direction, the equation to the curve is of the form

$$(x - a)\, y^2 = mx^2 (x - b).$$

Take the height of the tetrahedron as 80 units, we find that the curve cuts the vertical through A at a distance of $\frac{9}{10} \times 80 = 72$; also the curve cuts the transversals at (40, 33), and the edges of the tetrahedron at (80, 33), for the edge of the tetrahedron is very nearly 99. The equation reduces approximately to

$$(x - 60)\, y^2 = \cdot 4254 x^2 (x - 7),$$

$$\therefore\ x = -16,\ 0,\ 12,\ 16,\ 24,\ 31,\ 40,\ 48,\ 56,\ 60,\ 72,\ 76,\ 80,$$

$$\pm y = \quad 9,\ 0,\ \ 9,\ 12,\ 18,\ 24,\ 33,\ 44,\ 52,\ \infty,\ \ 0,\ 25,\ 33.$$

Intermediate points can be obtained from the equation.

Next take a vertical section through A and B, which also passes through the centre of the tetrahedron and through the point P of figure 15, and is represented by figure 16.

Take the axis of x again vertical through A. Then as before AP = 72.

AB is the position of the edge of a regular tetrahedron where AP is the perpendicular through the vertex to the base. Its position can be found by geometrical construction, or we may take the coordinates of B as very approximately (80, 56·6), taking $40\sqrt{2} = 56{\cdot}6$. It is seen by inspection that the curve completing the section is a conic that has horizontal tangents when $x = 40$ and when $x = 72$. That is at the plane of the transversals and at P.

It also passes through B and the origin.

The equation to the conic is then

$$10x^2 - 10xy\sqrt{2} - 4y^2 = 720(x - y\sqrt{2}).$$

From this equation the hyperbola can be drawn.

Now to construct the model take a frame made by joining corresponding angles of two parallel, equal, equilateral triangles.

The lower of these should be drawn, but not cut out, upon a piece of wood, rather larger than the triangle. Then a piece should be cut out as indicated by the figure *lpnrmq* in figure 13, for we need a hole beneath to work the under part of the model. The positions of the points l, m, n, are found from figure 16, m and n being symmetrically placed at the same distance from the centre as l, along bisectors of the angles of the triangle.

Only part of the base triangle is drawn.

Next cut a figure out of a thin piece of wood of the shape *abcdefgtz* shewn in figure 13. This is not part of the base, but is shewn on figure 13, it is to be placed in a plane parallel to the base as shewn by the letters *az* in figure 14.

Now take the upper equilateral triangle; this is to be fixed by three uprights as HF and KG (figure 14), so that its angular points are vertically above those of the base at such a height that the sections 15 and 16 exactly take up their proper positions.

Before being placed in position the points L, M, N should be found and holes bored for wires.

OL = OM = ON measured along the bisectors of the angle and OL is taken from figure 16, O being the point at which the vertical from A meets the horizontal through L (figure 16).

Figure 14 represents one of three equal vertical faces each containing a side of 12 and 13 as FG and HK. The side RS would be pulled in by the wires attached to it, but this is prevented by the piece *abcdefgtz*, which is placed in a horizontal position so that the sides *az*, *cd*, *fg*, fit against the sides such as RS and are secured to them by small screws at *a*, *z*, *c*, *d*, *f* and *g*. If the lower part of RS and of the other two corresponding faces is fastened to the base, the wires attached at R and S will only tend to keep them more firmly in position. We also find that *abcdefgtz* helps to support the plastocene of which the model is composed. Pins of the right length may be driven into it to mark the position of P and other points on the curves shewn by the figures 15 and 16.

There are of course three sections such as 15 or 16. To fix the wires representing the sides of the tetrahedron place Ll, Mm, Nn in position by passing through the holes L, M, N, and securing below at the points l, m, n. Then wires are attached at the points R, S and carried parallel to the sides of the base to corresponding points on the other faces.

The directions of these wires are shewn in figure 12. They are not in the plane of figure 12, but parallel to it, the true position of R and S being shewn in figure 14. These wires lie in the plane of B, C, D, and pass through these points as shewn in figure 12. They meet the wires Ll, Mm, Nn in B, C, D.

To place the wires representing the transversals we note first that their plane is halfway between A and the plane of B, C, D, and secondly that they form an equilateral triangle equal to BCD but inverted, as shewn by the figure *pqr* in figure 13; the middle points of the sides of this triangle lie on the edges of the tetrahedron.

Producing the sides of *pqr* to meet the sides of the base we obtain the points *h*, *k* (figure 13) and then measure the distance *hk* in figure 14, placing it so as to be bisected by the vertical through A, and halfway between A and RS. Having found *h* and *k* we find the corresponding points in the other three faces, and fix the transversals in position, as shewn by the triangle *pqr*.

The equations to the horizontal sections have been given on page 70. It has also been shewn there how to find them from the vertical section (figure 16).

Figure 15 gives additional points that help to verify results. The curve at T when produced forms a ridge of an elevated portion of the surface, while the curve at Q produced marks a line of greatest depression.

The curves at X produced approach to a horizontal asymptote.

The model of a surface having twenty-seven real straight lines is given in the next chapter (figures 6 to 11). If we suppose this model to become symmetrical, and three holes to close up to hollow nodes, while the base of the upper portion contracts to a solid conic node, we obtain a fair idea of the model of the surface under consideration.

The third case to be considered is that in which two nodes are on one side of the transversal plane, and two on the other. We find as we might expect two hollow nodes and two solid ones. If we suppose one line on the transversal plane to move to infinity, where the plane of the transversals is parallel to two opposite edges of the tetrahedron forming the nodes, and midway between them, we get a symmetrical surface.

However we do not get a cubic curve if we take planes parallel to the plane of the transversals, but a conic section. In this particular it is something like the first surface (page 71), but we note the following differences—A horizontal section (parallel to the plane of the transversals and at a considerable distance above it) consists of a solid ellipse, and as we come lower this form still continues to the edge AB (figure 22); we then get a straight line, or rather four coincident straight lines, for the minor axis of the ellipse vanishes; below AB to the centre O we have hyperbolic sections, which at the centre reduce to two straight lines (figure 19). But on page 73 the order of these sections was exactly reversed.

We take as the equation to the surface

$$z(x^2 + y^2 - z^2 + 1) = 2xy$$

referred to rectangular Cartesian coordinates. The origin is at O, and axis z vertical, the axes of x and y are the remaining transversal lines (figure 19). The coordinates of the nodes are $(\sqrt{2}, 0, 1)$, $(-\sqrt{2}, 0, 1)$, $(0, -\sqrt{2}, -1)$, and $(0, \sqrt{2}, -1)$.

First take two sections, vertical, through the nodes A, B and C, D respectively. As parts of these sections are the straight lines AB and CD the section is in each case completed by a symmetrical conic.

Figure 22 represents the line AB together with the corresponding hyperbola, and figure 20 is the section through CD.

The vertical section through a transversal is in either case represented by figure 21.

Just as on page 73 we used the parabola as guide for drawing horizontal sections, so here we use figures 19 to 22.

A model may be described. Cut out of a piece of wood a square hole (figure 18) and fix above it another square (figure 17) by uprights as indicated. The height of the uprights is shewn by the vertical sections. On the upper section mark the points a, b, c, and d, and on the lower one a', b', c', d'.

aa', bb', cc', dd' give the position of the edges of the tetrahedron, represented as before by wires.

a, b, c, d, and a', b', c', d' are found from the fact that they are the points at which the inscribed ellipses touch the squares. The axes of the ellipses XY, LM, PQ, lm are shewn in the sections 20 and 22.

The points E, e are taken from figure 21.

The wires representing the transversals (figure 19) are fixed at the middle points of the uprights at the sides of the squares (figures 17 and 18).

The edges of the tetrahedron should meet at A, B, C, and D, and they should be intersected also by the transversals, except the edges AB and CD, which lie in parallel planes. The general

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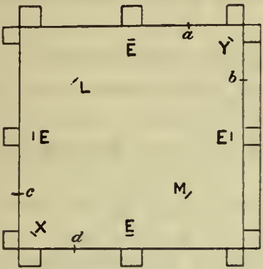

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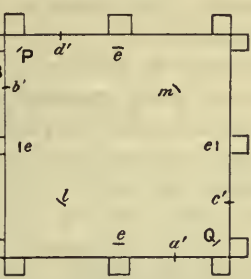

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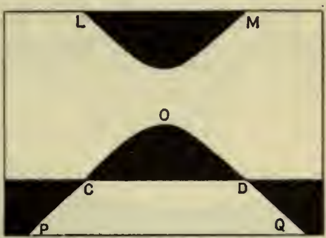

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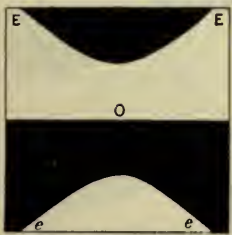

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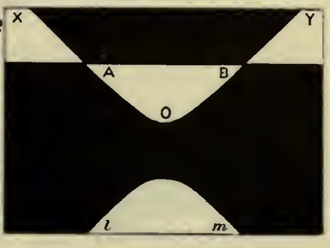


appearance of the surface is then above AB a conical solid every transverse section of which is an ellipse, and below CD a hollow cone, every section of which is also an ellipse, but the minor axes of the lower set of ellipses lie in the same direction as the major axes of the upper, and *vice versâ*. (See figures 17 and 18.) The lower square was cut out in figure 18 to work the model from beneath. The nodes at A and B are solid nodes and at C and D hollow.

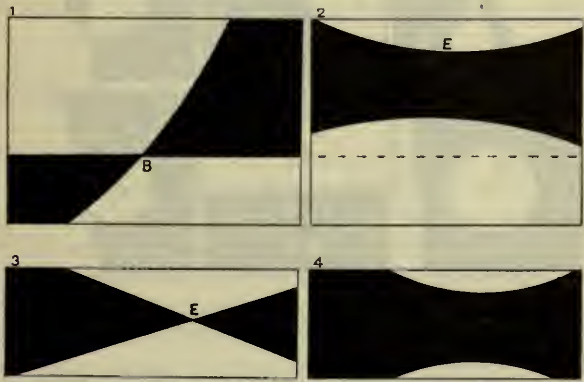
The axes of the horizontal sections, whether ellipse or hyperbola, are very easily expressed from the equation to the surface, for if h be the height of any section above the centre, the squares on the semi-axes are $h+1$ and $h-1$ above AB, and $h+1$ and $1-h$ between AB and O.

Every section below O is exactly equal to that at the same distance above it, but turned through a right angle, and solid and hollow interchanged. The infinite sheet between AB and CD has *two* elevations and *two* depressions, not *three* as before; this is because a transversal is projected to infinity. Compare Chapter VI. (figures 12 and 13).

CHAPTER VI.

ON MODELS OF CUBIC SURFACES HAVING TWENTY-SEVEN REAL STRAIGHT LINES.

It was stated that Zeuthen had proved by the theory of continuity that the forms of cubic surfaces having twenty-seven real straight lines may be derived from those having four conic nodes. Now taking the model described on page 73, and varying the equation to the surface by adding terms having small numerical coefficients, we obtain a surface somewhat similar in shape but no longer having nodes at A, B, C, and D.



Compare the node at B (figure 8, Chapter v.), which is again shewn by figure 1 in this chapter, with the sections shewn by figures 2, 3, 4.

When the terms are added to the equation the section at B takes the shape of figure 2, the horizontal section at E being two straight lines.

We move the horizontal section lower, it takes the shape of 4, and then again still lower that of another section shewn by 3.

Next we probably find the hyperbola changes to a parabola, and the parabola to an ellipse.

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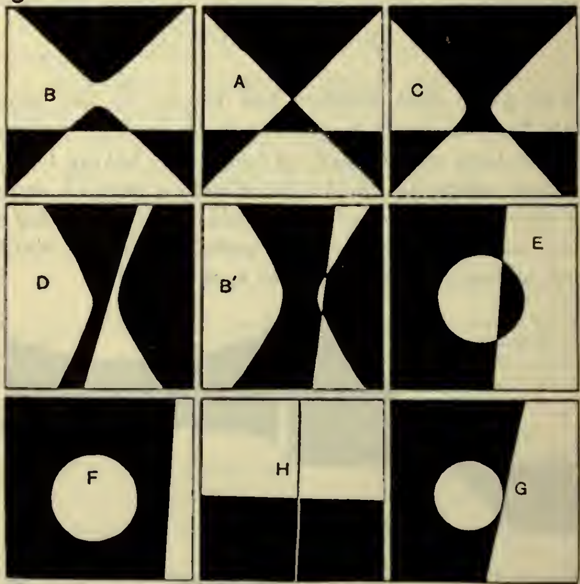

In these changes the transversal is still supposed at infinity or at a very great distance.

If this is not the case the sections must be taken as passing through the transversal, and not parallel.

We find then that the four coincident straight lines of figure 1 are now replaced in the general case by two pairs of straight lines and a parabola. If the transversal is not at infinity we usually find two parabolas.

Taking the sections about each transversal for each node we find twenty-four straight lines, a double six for each node. See page 22. These with the transversals complete the twenty-seven lines.

We need to examine the shapes of sections of the surface, in the general case, when a plane is drawn through a straight line on the surface. These are shewn in 5.

A circle is drawn in E, F, G; it may be replaced by an ellipse, and in B or C one of the infinite branches of the hyperbola may move to infinity, leaving a straight line and a parabola.

Again, solid and hollow portions may be interchanged. Thus in A we may have a hollow triangle with hollow branches at the vertices, the remainder being solid.

It is convenient to call such a section A′, with similar notation for the other sections.

The section H needs special notice. It is not, strictly speaking, a section of the general case, for it represents a transversal crossing coincident lines. The meaning of "*transversal*" is clearly seen, for the surface crosses the coincident lines at the transversal, being represented as hollow on one side of it and solid on the other.

The transversal may itself coincide with the coincident lines, but though apparently the same section, this case must not be confused with that in which the transversal moves to infinity.

We note also that in A the area of the triangle may become zero, this does not cause any singularity in the surface.

Comparing the equation on page 24 with the model, page 73, we see that the diagonal surface corresponding to the four conic node surface, having its transversal plane at infinity, has for its equation

$$x^2 + y^2 + z^2 + 2xyz - 5 = 0.$$

Three straight lines on the surface are in the plane at infinity.

The shape of the surface is that of a central solid with four infinite branches each symmetrical about straight lines drawn from the origin to the points at which the nodes were situated in the

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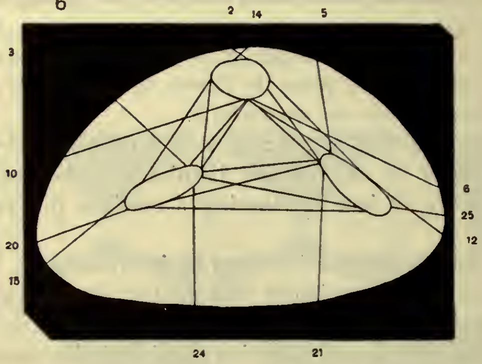

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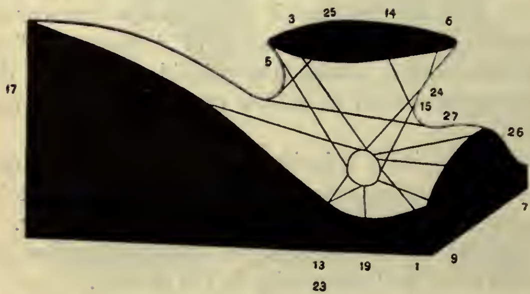

corresponding nodal cubic. The surface about these points has assumed the shape described on page 85. See also page 71.

The twenty-four remaining straight lines lie by pairs in the planes $z=1$, $z=-1$, $z=\sqrt{5}$ and $z=-\sqrt{5}$, with corresponding planes for the other axes.

The pentahedron has for its faces the plane at infinity, and the planes $x+y+z-1=0$, $x-y-z-1=0$, $-x-y+z-1=0$ and $-x+y-z-1=0$. (See page 24.)

We must take the constant 4 as the equation to the plane at infinity, so that the sum of these equations may be identically

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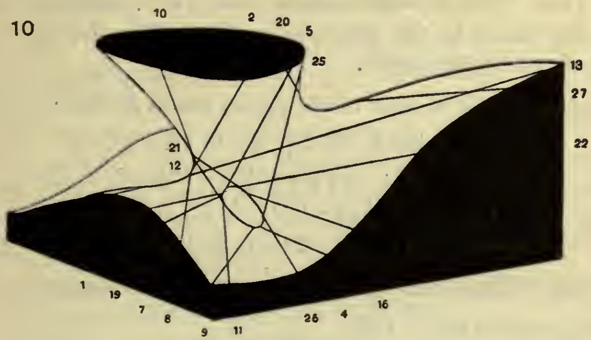

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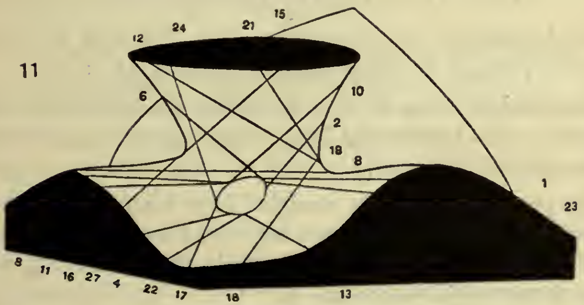


zero. The sum of their cubes will then reduce to the given equation.

It has been mentioned in the last chapter that the surface having a solid conic node and three hollow ones can be derived from the model of a surface shewn by the figures 6 to 11.

In this model all the twenty-seven straight lines are real. The position of these lines was first found by adjustment, and verified by equations. Horizontal sections were built up by layers of wood rather smaller than the actual size. The whole was then covered with plaster of Paris to measurement, and the lines placed upon the model.

In order to use rectangular coordinates conveniently, the lines 4 and 13 were taken as axes $x=0$, $z=0$; and $y=0$, $z=0$; and the line 9 in the plane of $z=0$ cutting the other axes $12\frac{1}{2}$ inches from the origin.

The equations of some of the planes are as follows:

$$x=0 \text{ contains } 4,\ 12,\ 2;\quad y=0,\ 13,\ 14,\ 15;$$
$$z=0,\ 4,\ 9,\ 13;\quad x+y+(4\tfrac{1}{6})\,z-12\tfrac{1}{2}=0,\ 9,\ 7,\ 8;$$
$$11x+10y+(27\tfrac{1}{2})\,z-110=0,\ 12,\ 7,\ 15;$$

and

$$10x+14y-(12\tfrac{1}{2})\,z-24=0,\ 2,\ 14,\ 8.$$

Then we may write down the equation to the surface as

$$\begin{aligned} &\mathsf{M}xy\,(x+y+(4\tfrac{1}{6})\,z-12\tfrac{1}{2}) \\ &\qquad = z\,(11x+10y+(27\tfrac{1}{2})\,z-110)\,(10x+14y-(12\tfrac{1}{2})\,z-24). \end{aligned}$$

The heights of the intersections (14, 25), (20, 2), (12, 15) above the plane of $z=0$ are in the proportion 3 : 2 : 1.

The straight lines 20 and 25 divide the line 9 intercepted between 4 and 13 internally in the proportion 25 : 2, 20 meeting it near 13, and 25 near 4.

The projections of the straight lines 1 and 11 upon $z=0$ divide the line 9 as intercepted between 4 and 13, externally in the proportion 1 : 6, and 4 internally in the ratio 1 : 2, 1 being nearer to the origin on the line 4.

By drawing any triangular section as (4, 9, 13) and marking the points we know, and using the rule that three points at which any plane of the table of reference meets it intersect 4, 9, and 13 in a straight line, we find points upon the remaining lines.

To find the value of M we take the equation to the line 1 as found from the condition that it passes through the point

$(-2\frac{1}{2}, 15, 0)$ and meets the lines 15 and 2, the equations to which are

$$y=0,\ x+(2\tfrac{1}{2})z-10=0;\ \text{and}\ x=0,\ 14y-(12\tfrac{1}{2})z-24=0.$$

We thus find the equation to the line 1 as

$$x+2\tfrac{1}{2} : 545 :: y-15 : -2382 :: z : 576.$$

Substituting $542\frac{1}{2}$, -2367, 576 for x, y, z in the equation to the surface, for every point of 1 is on the surface, we find the value for M.

The above method was indicated on page 25.

It is simpler, however, in this case to find the equations to straight lines from those already found, and not from that of the surface, or better still on a model to find the sections by drawing, and test by analysis.

In this model the lines were so adjusted that the plane $x+y=12\frac{1}{2}$ is the plane of the lines 9, 20, 25.

Also 20 and 25 meet the planes $y=0$ and $x=0$ where they meet 2 and 14, which are known points.

The form of the surface at infinity approximates to the cubic cone which has been already described in Chapter IV.

Figure 6 shews the under part of the model with three holes, which in the model (page 78) contract to three hollow nodes.

Figures 7 and 8 are vertical sections, to a smaller scale, to illustrate the existence of a hole.

As section 7 revolves, we get a nodal cubic, then figure 8, then a nodal cubic again, and the figure 7 again represents the section.

Figures 9, 10, 11 are views of the model standing upon a table, and turned round in order to shew from above the holes of figure 6.

We observe that a double six passes through each hole, also between each hole and the next.

A double six, 3, 25, 14, 6, 15, 10, 2, 20, 5, 24, 12, 21, passes upwards along the central conical portion from which the oval is cut at infinity (Chapter IV.).

The same lines, as we should expect, also are continued downwards along the hollow conical portion below.

We can reduce the surface to one having conic nodes in several ways, for we have the choice of making an opening close up, or of making the solid portion between two holes contract, and thus obtaining a solid node.

For example, let the lines 1, 13; 14, 6; 4, 11; 22, 16; 23, 17; 10, 2 coincide by pairs (page 9).

The hole in figure 11 closes up, and we obtain a conic node.

It is easy to see that the other holes may close up, the lines coinciding as shewn in the table of reference for two and three conic nodes.

When we come to the fourth node it is necessary to suppose the oval shewing the upper part of the model in the figures 9, 10, 11 to contract, we thus get a fourth node.

The surface corresponds to that described in the last chapter, of three hollow and one solid conic node.

The lines 8, 18, 27 are the transversals.

If we let a conic node move up and coincide with another part of the surface we obtain a binode. A binode does not stand out from the rest of the surface appearing as the vertex of a cone of the second degree, but causes what seems to be a fold in the surface.

The transverse section of a binode B_3 is given in the next chapter, figure 4. The cusp shews a wedge-shaped portion of the surface. There is also a model of the binode B_5.

The reason for this form of the surface is given in Chapter VII. The cone shape appears when an even number of plane triangular sections coincide.

Lastly, we take a model of an irregular shaped cubic surface in which two nodes are above the transversal plane and two below it.

The conic node rays AB, AC, AD, BC, BD, CD joining the four nodes A, B, C, D must be represented by the coincidence of twenty-four straight lines as follows :—

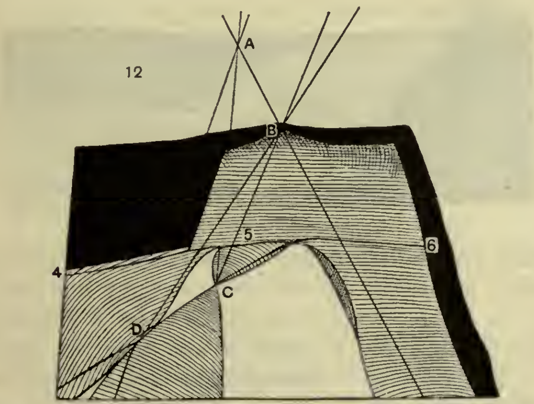
23, 1, 15, 24; 14, 21, 18, 11; 12, 27, 16, 2; 17, 9, 13, 26; 3, 19, 20, 7; and 10, 22, 25, 8.

The remaining lines 4, 5, 6 are the transversals.

Take Cartesian coordinates, the lines 4, 5, 6 being the intersections of the three planes $x=0$, $2x+y=6$, and $y=0$ with $z=0$, the equations to the planes of the tetrahedron ABCD being

$$\begin{aligned}\tfrac{1}{6}x+\tfrac{1}{5}y+\tfrac{11}{30}z-1&=0,\\ \tfrac{1}{6}x+\tfrac{2}{15}y+\tfrac{3}{10}z-1&=0,\\ \tfrac{1}{2}x+\tfrac{1}{5}y-\tfrac{8}{15}z-1&=0,\\ \tfrac{1}{2}x+\tfrac{2}{15}y+\tfrac{3}{10}z-1&=0.\end{aligned}$$

Suppose, then, the surface to be represented by cutting away portions of a solid block, it will be found impossible to represent the whole of the surface without causing solid portions to hide other parts from view.



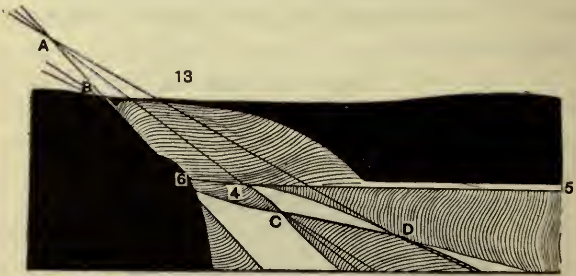
In the photographs of the model which are here given, much of the surface has been cut away to allow of the nodes B, C, D being seen, while the part above B has not been constructed.

The horizontal section of the surface below D consists of an oval and three infinite branches. The oval has upon it four given points, namely the intersections of the plane with the node lines CA, CB, AD, and BD, but AB and CD meet it on two of the infinite branches.

The horizontal section through D has a node at D in which the oval joins one of the infinite branches.

Horizontal sections, below C but above D, again consist of an oval and three infinite branches. The three conic node rays AC, BC, and DC meet these sections in the oval, the remaining rays meeting it on the infinite branches.

Above C, but below the plane of the transversals, the horizontal sections are of the same kind as those between C and D, except that, as the sections approach the plane of the transversals, the shape of the section approximates to three straight lines.



When the horizontal section coincides with the plane of the transversals, it consists of the three straight lines numbered 4, 5, 6; 4 is seen on figure 12, it passes under a kind of saddle soon after meeting BD, it meets 5 and then AC as seen in figure 13, it next cuts 6 passing under another saddle-shaped portion of the surface.

5 may be traced by first taking figure 13; it meets AD, then 4, next BC (figure 12); after meeting 6 it passes to the back of the solid portion on the right of figure 12.

6 is only seen on figure 12. The number 6 on figure 13 indicates the point at which it cuts the section, for in figure 13

the line 6 slopes inwards to the left and is completely hidden by the surface: it meets AB and CD.

We observe that in horizontal sections the tangents at the points in which AC and BD meet them are parallel to 4, at the points in which AD and BC meet them are parallel to 5, and at the points in which AB and CD meet them are parallel to 6, except at the nodes.

Suppose we denote the triangle formed by 4, 5, 6 by the letters L, O, M.

Produce LM to E, ML to F, OL to G, LO to H, OM to K, and MO to Q.

In a section parallel to the plane of the transversals, and just below it, the section consists of an oval nearly coincident with the triangle LOM, and three infinite branches nearly coincident with the straight lines containing the angles QOH, KME and GLF.

In a section just above the plane of the transversals, we have three infinite branches nearly represented by EM, MO, OH; KM, ML, LG, and FL, LO, OQ.

Above the plane of the transversals the horizontal sections are generally similar in character to those below but in the reverse order, and solid portions are now represented by corresponding hollow portions; for example, between C and D as we have seen, horizontal sections consist of an oval cut from a conical solid and three infinite branches cut from three infinite solid portions, but between A and B we find a hollow oval, and three infinite hollow branches.

The only exception is that part of the surface immediately above the plane of the transversals, but below B. This part of the surface when cut by horizontal sections gives three infinite branches but no oval.

CHAPTER VII.

ON NODES.

It has been shewn that a surface generated by three projective pencils, by the intersection of three corresponding planes, is such that if any straight line lies on the surface, any plane section through the line consists of this straight line and a conic. We find also that if four points on the surface lie on a straight line, then this straight line lies entirely in the surface. If the projectivity of the pencils be determined by four non-intersecting straight lines, these lines obviously lie on the surface.

Reye's methods do not lead to the consideration of conic nodes, for he is careful to state that the six non-intersecting lines of a double six cannot meet a plane on a conic, which of course is true provided the lines do not intersect. But if they meet at a point we get a conic node, and the lines do meet any plane on a conic.

We could however define a cubic surface as one on which a certain number of straight lines lie (suppose we say three straight lines which form a plane triangle), and is such that every section through a straight line is made up of this line and a conic. It would then be necessary by Reye's methods to shew that there is such a surface.

This could be done, as on pages 31 to 48.

By taking five straight lines that meet one straight line, but which do not themselves intersect, and through them drawing a cubic surface (p. 40), we find the complete scheme of lines on the surface. The converse of this proposition, that any cubic surface has such a scheme of lines, involves Reye's proof (p. 43).

Next, we may state that either a double six or the remaining fifteen lines are sufficient to determine the surface.

This is evident either by analysis or geometrically: (1) by analysis, for on pages 4 and 25 it was shewn that nine of these lines (in the case of the fifteen), together with a point on a tenth, give all the necessary constants; and (2) geometrically, because any plane section through one of the lines meets more than five lines on the surface independently of the first, and these points fix the conic that completes the section.

Suppose then we have fifteen straight lines that intersect one another according to the table of reference of the lines 1 to 15.

It is evident that when we come to draw two straight lines to meet four of these lines that do not intersect (see p. 50), these lines may become coincident.

Again, if we draw any plane to intersect the system of lines, and remember that three straight lines in a plane cut this plane in three points in a straight line, the coincidence of two such lines is found to involve the coincidence of any pair drawn to meet any set of four non-intersectors.

Further, as on page 56, we can shew that these coincident pairs form six rays that lie on a cone. The surface about the point at which these six rays meet is clearly of the shape of a quadric cone. This is also evident from the coincidence of sections about a line that form a plane triangle. (See page 100.)

Next, no conic node can occur except at the intersection of straight lines on the surface, for we may take it as evident that every plane section through a conic node has a node at that point. Take any straight line on the surface which does not pass through the node, and draw a plane through this line and the conic node. Then the conic that with the straight line makes up the section has a node at the conic node. But the only conic that has a node is a pair of straight lines, therefore the section must consist of three straight lines, two of which intersect at the conic node.

If we look at the model of any cubic surface having twenty-seven real straight lines, we note the results otherwise obtained by Zeuthen's scheme (p. 22).

There are four holes or openings; also if we suppose every part of the model that is hollow to become solid, and *vice versâ*, we still have four holes lying between the former ones.

Through each hole, in either case, passes a double six.

The first kind of alteration that may take place is the contraction of a hole to a point. Hence we find, as before, that a double six passes through a point. We next take a second node due to the contraction of another hole, and in the same way arrive at the cases of three and four conic nodes.

Next, suppose that in the case of one conic node, the node moved up to one of the adjoining portions of the surface. Then the node disappears as a conic node, and its rays must become coincident with lines on the adjoining portion.

Comparing this result with the table of reference, we see that if any rays of a conic node coincide with any of the remaining lines, *e.g.* r_3 with 1, then the triangle $r_1 r_2 1$ must become $r_1 r_2 r_3$.

Therefore three rays lie in one plane.

Comparing this with the results obtained by taking plane sections through a straight line we find, as before, that for a conic node two sections have coincided, and for a binode three have become coincident.

If three rays lie in one plane, and we have shewn that the six rays lie on a cone, it is clear that any transverse section of this cone has degenerated into two straight lines, and the remaining three rays lie in a plane.

This agrees with what we know of the binode B_3.

Next let the part of the surface upon which the binode lies contract to the shape of the vertex of a cone. It is evident that we obtain the same result by first supposing that we have two conic nodes and letting these coincide.

The table of reference reduces to that given for B_4, while the model shews that one hole has disappeared.

The changes for B_5 can be found exactly in the same way, by supposing B_4 to move up to another part of the surface, and the conical form disappears.

B_6 may be derived from B_5 as B_4 was from B_3, or otherwise by the coincidence of three conic nodes, and another hole has disappeared.

It has been shewn (p. 5) that we usually have five plane sections passing through a straight line that degenerate into plane triangles.

Further, if two of these sections move up to one another and coincide, the vertex of the triangle (the given line being considered as base) becomes a conic node (p. 10).

Suppose a third triangular section to move up to and coincide with the two that have already coincided, then any section taken first in one direction and then in the other about the line near the node, obeys the same law as if there had been no coincidence.

In other words, in one direction we have sections such as 5C, and in the other 5B (Chapter VI.) for a small displacement. The surface about the vertex does not now approximate to a cone of the second degree.

If, however, a fourth section moves up to and coincides with the other three, it is clear that the sections now taken through a small angle in either direction are of the same form, that is either both as 5C or 5B.

The surface again is conical in form about the vertex.

The following scheme will shew this more clearly. If we neglect other changes such as from parabola to ellipse, regarding only the changes to sections of the form of plane triangles, the order must be CABACAB .... (See figure 5, Chapter VI.)

Now if the first and second triangular sections A coincide, the B between them disappears; we then have CACAB .... The form of the section taken in either direction through a small angle about the line is of the form C, therefore it is evident that the shape of the surface about the vertex of the triangle is like that of the vertex of a cone of the second degree.

Let this triangular section coincide with the third A, the C between them disappears, and we have CAB .... If the section now revolves about the line, the order of sections is the same as at first.

The coincidence of lines on the surface always implies coincidence of sections. This will be seen from the tables of reference.

The number of nodes that can exist upon the surface, with all possible combinations of them, may be found from the equation to the surface by considering the number of constants at our disposal, but there is an interesting way of doing this from the table of reference, only allowing that not more than three straight lines may lie in one plane section of the surface, and that not more than two intersecting straight lines may meet four non-intersectors.

These coincidences have been traced in forming the different tables of reference in Chapter I.

If we attempt to let lines coincide in any other order, we obtain no new results but some of the series may be omitted.

When the straight lines on the surface become imaginary, the effect on a model of the surface is generally to diminish the number of holes or openings, and to cause parts of the surface that extended to infinity when the lines were real to become limited in extent. (See p. 22.)

To illustrate this, take a cubic surface that approximates to a plane and a sphere, for example one the equation to which is

$$z(x^2+y^2+z^2-a^2)+\mathrm{A}x+\mathrm{B}y+\mathrm{C}z+\mathrm{D}=0,$$

where A, B, C, D are small.

The sections through the centre of the sphere and perpendicular to the plane $z=0$, very nearly approximate to 5E

(Chapter VI.), with an opening in some positions as in figure 8 (Chapter VI.). But comparing with these figures 7 and 8 (Chapter VI.), instead of an infinite conical solid above and an infinite conical hollow below, we find the upper solid portion is approximately a solid hemisphere, and the lower portion a hollow hemisphere.

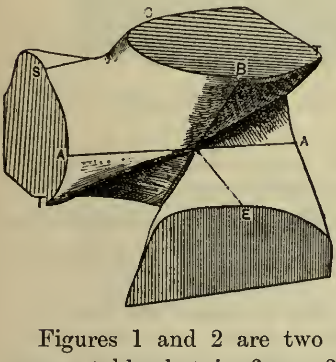
To compare with surfaces of the second degree, hyperboloids become ellipsoids when straight lines on the surface become imaginary.

However a cubic surface cannot be limited in every direction, for every plane section is a plane cubic curve, and every straight line meets a cubic curve in one real point, including points at an infinite distance. For instance, in the above surface we have an infinite sheet with elevations and depressions, but very nearly coinciding with $z=0$.

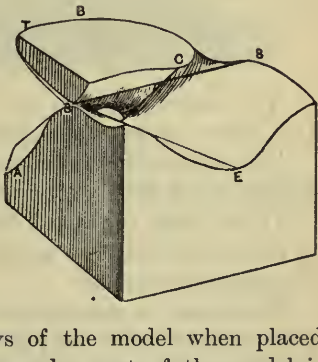
What we do not get at infinity comparing with the models in Chapter VI. where the lines are real, is the central ovals, the hemispherical portions being limited in extent.

A model is here given of a surface having a binode B_5. The lines on it correspond to the table of reference on page 12.

1



2

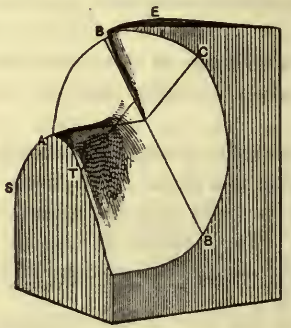


Figures 1 and 2 are two views of the model when placed upon a table, but in figure 3 the under part of the model is shewn.

In figure 2 we see the remaining hole.

Figures 1 and 3 shew the fold of the surface at the binode, so that plane cubics through the binode have cusps at the node.

3



A good plaster model of a surface having three binodes B_3 is exhibited at the South Kensington Museum.

Its equation is

$$(x+y+z-a)^3 = xyz$$

in rectangular Cartesian coordinates.

The central vertical line is $x=y=z$.

The nodes are the intersection of the axes with $x+y+z-a=0$.

The section (figure 4) is a vertical section through one of the axes, shewn by the dotted line.

The horizontal line through the node in the figure bisects the straight line joining the other two nodes, on the surface.

On the right the curve continually ascends, marking the line of greatest elevation of a ridge, while on the left it marks a line of greatest depression.

The surface is symmetrical, so that horizontal sections are all of the form described on page 70.

Above the point at which the vertical through the origin cuts the curve (figure 4), a horizontal section has no oval, it only has three infinite solid hyperbolic branches cut from the elevated portions.

When the section plane moves lower, we have an oval and three infinite solid branches till we reach the plane of the nodes.

There the section is triangular, as 5A, Chapter VI., the infinite branches becoming connected with the central oval (now a triangle).

Below the plane of the nodes we have three hollow hyperbolic branches but no oval.

As the surface is symmetrical, the equations to all horizontal sections are of the form given on page 68.

4

We find that they are

$$3(x-p)y^2 = x^3 + 3x^2p - 4p^3 + 108(p-l)^3,$$

where p is found by dividing by $\sqrt{2}$ the height of the section above the original origin, which is at a depth below the plane of the nodes $= a \div \sqrt{3}$.

We find l from $a = l\sqrt{6}$.

If we express the equation to the section in figure 4 in Cartesian coordinates, the node being the origin and axes horizontal and vertical, we find the values of the ordinate y involve

$\sqrt{2}$, but if we suppose the values of y divided by $\sqrt{2}$, the equation becomes

$$(x-2l)^2(x+3y+l)-4(y+l)^3+108y^3=0.$$

To trace the curve, then, we find a sufficient number of values of y from this equation corresponding to given values of x, and multiply the results by $\sqrt{2}$ to find the true ordinates.

When $y=0$, $x=3l$; when $x=0$, $y=\frac{3}{26}l$; when $x=2l$, $y=\frac{1}{2}l$; and when $y=-l$, $x=2l+3l\sqrt[5]{2}$.

[The results for y are to be multiplied by $\sqrt{2}$.]

The coordinates $x=2l$, $y=\frac{1}{2}l\sqrt{2}$ mark the highest point of the central portion, vertically above the original origin; its coordinates being $2l$, $-l\sqrt{2}$.

The surface consists of one infinite sheet having three elevated ridges with a corresponding depressed portion between each. These may be seen, though not symmetrical, in the figure of the second sheet of the cone at the commencement of Chapter IV. Instead of running in to the centre, as in the cone, the elevated portions terminate at the nodes as shewn in figure 4, while the depression opposite begins at the middle point of the opposite side.

The central portion is somewhat like a spherical triangle having the nodes at the angles.

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